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Applied Intertemporal Optimization

Edition 1.2 plus: Textbook and Solutions Manual

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I also profited from many discussions with economists and mathematicians from many other places. The high download figures at repec.org show that the book is also widely used on an international level. I especially would like to thank Christian Bayer and Ken Sennewald for many insights into the more subtle issues of stochastic continuous time processes. I hope I succeeded in incorporating these insights into this book. I would also like to thank MacKichan for their repeated, quick and very useful support on typesetting issues in Scientific Word.

Edition 1.1 from 2011 is almost identical from the first edition in 2010. Some typos were removed and few explanations were improved. Edition 1.2 exists only as edition 1.2 plus: Textbook and Solutions Manual. Apart from some minor changes as compared to edition 1.1., it includes solutions to selected problem sets as described at the beginning of the Solutions Manual on p. 321.

Mainz, August 2012

Overview

The basic structure of this book is simple to understand. It covers optimization methods and applications in discrete time and in continuous time, both in worlds with certainty and worlds with uncertainty.

	discrete time	continuous time
deterministic setup	Part I	Part II
stochastic setup	Part III	Part IV

Table 0.0.1 *Basic structure of this book*

		Solution methods	
Parts and chapters		Substitution	Lagrange
Ch. 1	Introduction		
Part I Deterministic models in discrete time			
Ch. 2	Two-period models and difference equations	2.2.1	2.3
Ch. 3	Multi-period models	3.8.2	3.1.2, 3.7
Part II Deterministic models in continuous time			
Ch. 4	Differential equations		
Ch. 5	Finite and infinite horizon models		5.2.2, 5.6.1
Ch. 6	Infinite horizon models again		
Part III Stochastic models in discrete time			
Ch. 7	Stochastic difference equations and moments		
Ch. 8	Two-period models	8.1.4, 8.2	
Ch. 9	Multi-period models	9.5	9.4
Part IV Stochastic models in continuous time			
Ch. 10	Stochastic differential equations, rules for differentials and moments		
Ch. 11	Infinite horizon models		
Ch. 12	Notation and variables, references and index		

Table 0.0.2 *Detailed structure of this book*

Each of these four parts is divided into chapters. As a quick reference, the table below provides an overview of where to find the four solution methods for maximization problems used in this book. They are the “substitution method”, “Lagrange approach”, “optimal control theory” and “dynamic programming”. Whenever we employ them, we refer to them as “Solving by” and then either “substitution”, “the Lagrangian”, “optimal control” or “dynamic programming”. As differences and comparative advantages of methods can most easily be understood when applied to the same problem, this table also shows the most frequently used examples.

Be aware that these are not the only examples used in this book. Intertemporal profit maximization of firms, capital asset pricing, natural volatility, matching models of the labour market, optimal R&D expenditure and many other applications can be found as well. For a more detailed overview, see the index at the end of this book.

		Applications (selection)			
optimal control	Dynamic programming	Utility maximization	Central planner	General equilibrium	Budget constraints
		2.1, 2.2	2.3.2	2.4	2.5.5
	3.3	3.1, 3.4, 3.8	3.2.3, 3.7	3.6	
					4.4.2
5		5.1, 5.3, 5.6.1	5.6.3		
	6	6.1		6.4	
		8.1.4, 8.2		8.1	
	9.1, 9.2, 9.3	9.1, 9.4	9.2		
					10.3.2
	11	11.1, 11.3		11.5.1	

Table 0.0.2 *Detailed structure of this book (continued)*

Contents

Chapter 1

Introduction

This book provides a toolbox for solving dynamic maximization problems and for working with their solutions in economic models. Maximizing some objective function is central to Economics, it can be understood as one of the defining axioms of Economics. When it comes to dynamic maximization problems, they can be formulated in discrete or continuous time, under certainty or uncertainty. Various maximization methods will be used, ranging from the substitution method, via the Lagrangian and optimal control to dynamic programming using the Bellman equation. Dynamic programming will be used for all environments, discrete, continuous, certain and uncertain, the Lagrangian for most of them. The substitution method is also very useful in discrete time setups. The optimal control theory, employing the Hamiltonian, is used only for deterministic continuous time setups. An overview was given in fig. 0.0.2 on the previous pages.

The general philosophy behind the style of this book says that what matters is an easy and fast derivation of results. This implies that a lot of emphasis will be put on examples and applications of methods. While the idea behind the general methods is sometimes illustrated, the focus is clearly on providing a solution method and examples of applications quickly and easily with as little formal background as possible. This is why the book is called *applied* intertemporal optimization.

- Contents of parts I to IV

This book consists of four parts. In this first part of the book, we will get to know the simplest and therefore maybe the most useful structures to think about changes over time, to think about dynamics. Part I deals with discrete time models under certainty. The first chapter introduces the simplest possible intertemporal problem, a two-period problem. It is solved in a general way and for many functional forms. The methods used are the Lagrangian and simple substitution. Various concepts like the time preference rate and the intertemporal elasticities of substitution are introduced here as well, as they are widely used in the literature and are used frequently throughout this book. For those who want to understand the background of the Lagrangian, a chapter is included that shows the link between Lagrangians and solving by substitution. This will also give us the opportunity to explain the concept of shadow prices as they play an important role e.g. when using Hamiltonians or dynamic programming. The two-period optimal consumption setup will then be put into a decentralized general equilibrium setup. This allows us to understand general equilibrium structures in general while, at the same time, we get to know the standard overlapping generations (OLG) general equilibrium model. This is one of the most widely used dynamic models in Economics. Chapter 2 concludes by reviewing some aspects of difference equations.

Chapter 3 then covers infinite horizon models. We solve a typical maximization problem first by using the Lagrangian again and then by dynamic programming. As dynamic programming regularly uses the envelope theorem, this theorem is first reviewed in a simple static setup. Examples for infinite horizon problems, a general equilibrium analysis of a decentralized economy, a typical central planner problem and an analysis of how to treat family or population growth in optimization problems then complete this chapter. To complete the range of maximization methods used in this book, the presentation of these examples will also use the method of "solving by inserting".

Part II covers continuous time models under certainty. Chapter 4 first looks at differential equations as they are the basis of the description and solution of maximization problems in continuous time. First, some useful definitions and theorems are provided. Second, differential equations and differential equation systems are analyzed qualitatively by the so-called "phase-diagram analysis". This simple method is extremely useful for understanding differential equations per se and also for later purposes for understanding qualitative properties of solutions to maximization problems and properties of whole economies. Linear differential equations and their economic applications are then finally analyzed before some words are spent on linear differential equation systems.

Chapter 5 presents a new method for solving maximization problems - the Hamiltonian. As we are now in continuous time, two-period models do not exist. A distinction will be drawn, however, between finite and infinite horizon models. In practice, this distinction is not very important as, as we will see, optimality conditions are very similar for finite and infinite maximization problems. After an introductory example on maximization in continuous time by using the Hamiltonian, the simple link between Hamiltonians and the Lagrangian is shown.

The solution to maximization problems in continuous time will consist of one or several differential equations. As a unique solution to differential equations requires boundary conditions, we will show how boundary conditions are related to the type of maximization problem analyzed. The boundary conditions differ significantly between finite and infinite horizon models. For the finite horizon models, there are initial or terminal conditions. For the infinite horizon models, we will get to know the transversality condition and other related conditions like the No-Ponzi-game condition. Many examples and a comparison between the present-value and the current-value Hamiltonian conclude this chapter.

Chapter 6 solves the same kind of problems as chapter 5, but it uses the method of "dynamic programming". The reason for doing this is to simplify understanding of dynamic programming in stochastic setups in Part IV. Various aspects specific to the use of dynamic programming in continuous time, e.g. the structure of the Bellman equation, can already be treated here under certainty. This chapter will also provide a comparison between the Hamiltonian and dynamic programming and look at a maximization problem with two state variables. An example from monetary economics on real and nominal interest rates concludes the chapter.

In part III, the world becomes stochastic. Parts I and II provided many optimization methods for deterministic setups, both in discrete and continuous time. All economic questions that were analyzed were viewed as "sufficiently deterministic". If there was any uncertainty in the setup of the problem, we simply ignored it or argued that it is of no importance for understanding the basic properties and relationships of the economic question. This is a good approach to many economic questions.

Generally speaking, however, real life has few certain components. Death is certain, but when? Taxes are certain, but how high are they? We know that we all exist - but don't ask philosophers. Part III (and part IV later) will take uncertainty in life seriously and incorporate it explicitly in the analysis of economic problems. We follow the same distinction as in part I and II - we first analyse the effects of uncertainty on economic behaviour in discrete time setups in part III and then move to continuous time setups in part IV.

Chapter 7 and 8 are an extended version of chapter 2. As we are in a stochastic world, however, chapter 7 will first spend some time reviewing some basics of random variables, their moments and distributions. Chapter 7 also looks at difference equations. As they are now stochastic, they allow us to understand how distributions change over time and how a distribution converges - in the example we look at - to a limiting distribution. The limiting distribution is the stochastic equivalent to a fix point or steady state in deterministic setups.

Chapter 8 looks at maximization problems in this stochastic framework and focuses on the simplest case of two-period models. A general equilibrium analysis with an overlapping generations setup will allow us to look at the new aspects introduced by uncertainty for an intertemporal consumption and saving problem. We will also see how one can easily understand dynamic behaviour of various variables and derive properties of long-run distributions in general equilibrium by graphical analysis. One can for example easily obtain the range of the long-run distribution for capital, output and consumption. This increases intuitive understanding of the processes at hand tremendously and helps a lot as a guide to numerical analysis. Further examples include borrowing and lending between risk-averse and risk-neutral households, the pricing of assets in a stochastic world and a first look at 'natural volatility', a view of business cycles which stresses the link between jointly endogenously determined short-run fluctuations and long-run growth.

Chapter 9 is then similar to chapter 3 and looks at multi-period, i.e. infinite horizon, problems. As in each chapter, we start with the classic intertemporal utility maximization problem. We then move on to various important applications. The first is a central planner stochastic growth model, the second is capital asset pricing in general equilibrium and how it relates to utility maximization. We continue with endogenous labour supply and the matching model of unemployment. The next section then covers how many maximization problems can be solved without using dynamic programming or the Lagrangian. In fact, many problems can be solved simply by inserting, despite uncertainty. This will be illustrated with many further applications. A final section on finite horizons concludes.

Part IV is the final part of this book and, logically, analyzes continuous time models under uncertainty. The choice between working in discrete or continuous time is partly driven by previous choices: If the literature is mainly in discrete time, students will find it helpful to work in discrete time as well. The use of discrete time methods seem to hold for macroeconomics, at least when it comes to the analysis of business cycles. On the other hand, when we talk about economic growth, labour market analyses and finance, continuous time methods are very prominent.

Whatever the tradition in the literature, continuous time models have the huge advantage that they are analytically generally more tractable, once some initial investment into new methods has been digested. As an example, some papers in the literature have shown that continuous time models with uncertainty can be analyzed in simple phase diagrams as in deterministic continuous time setups. See ch. 10.6 and ch. 11.6 on further reading for references from many fields.

To facilitate access to the magical world of continuous time uncertainty, part IV presents the tools required to work with uncertainty in continuous time models. It is probably the most innovative part of this book as many results from recent research flow directly into it. This part also most strongly incorporates the central philosophy behind writing this book: There will be hardly any discussion of formal mathematical aspects like probability spaces, measurability and the like. While some will argue that one can not work with continuous time uncertainty without having studied mathematics, this chapter and the many applications in the literature prove the opposite. The objective here is to clearly make the tools for continuous time uncertainty available in a language that is accessible for anyone with an interest in these tools and some "feeling" for dynamic models and random variables. The chapters on further reading will provide links to the more mathematical literature. Maybe this is also a good point for the author of this book to thank all the mathematicians who helped him gain access to this magical world. I hope they will forgive me for "betraying their secrets" to those who, maybe in their view, were not appropriately initiated.

Chapter 10 provides the background for optimization problems. As in part II where we first looked at differential equations before working with Hamiltonians, here we will first look at stochastic differential equations. After some basics, the most interesting aspect of working with uncertainty in continuous time follows: Ito's lemma and, more generally, change-of-variable formulas for computing differentials will be presented. As an application of Ito's lemma, we will get to know one of the most famous results in Economics - the Black-Scholes formula. This chapter also presents methods for how to solve stochastic differential equations or how to verify solutions and compute moments of random variables described by a stochastic process.

Chapter 11 then looks once more at maximization problems. We will get to know the classic intertemporal utility maximization problem both for Poisson uncertainty and for Brownian motion. The chapter also shows the link between Poisson processes and matching models of the labour market. This is very useful for working with extensions of the simple matching model that allows for savings. Capital asset pricing and natural volatility conclude the chapter.

- From simple to complex setups

Given that certain maximization problems are solved many times - e.g. utility maximization of a household first under certainty in discrete and continuous time and then under uncertainty in discrete and continuous time - and using many methods, the "steps how to compute solutions" can be easily understood: First, the discrete deterministic two-period approach provides the basic intuition or feeling for a solution. Next, infinite horizon problems add one dimension of complexity by "taking away" the simple boundary condition of finite horizon models. In a third step, uncertainty adds expectations operators and so on. By gradually working through increasing steps of sophistication and by linking back to simple but structurally identical examples, intuition for the complex setups is built up as much as possible. This approach then allows us to finally understand the beauty of e.g. Keynes-Ramsey rules in continuous time under Poisson uncertainty or Brownian motion.

- Even more motivation for this book

Why teach a course based on this book? Is it not boring to go through all these methods? In a way, the answer is yes. We all want to understand certain empirical regularities or understand potential fundamental relationships and make exciting new empirically testable predictions. In doing so, we also all need to understand existing work and eventually present our own ideas. It is probably much more boring to be hindered in understanding existing work and be almost certainly excluded from presenting our own ideas if we always spend a long time trying to understand how certain results were derived. How did this author get from equation (1) and (2) to equation (3)? The major advantage of economic analysis over other social sciences is its strong foundation in formal models. These models allow us to discuss questions much more precisely as expressions like "marginal utility", "time preference rate" or "Keynes-Ramsey rule" reveal a lot of information in a very short time. It is therefore extremely useful to first spend some time in getting to know these methods and then to try to do what Economics is really about: understand the real world.

But, before we really start, there is also a second reason - at least for some economists - to go through all these methods: They contain a certain type of truth. A proof is true or false. The derivation of some optimal behaviour is true or false. A prediction of general equilibrium behaviour of an economy is truth. Unfortunately, it is only truth in an analytical sense, but this is at least some type of truth. Better than none.

- The audience for this book

Before this book came out, it had been tested for at least ten years in many courses. There are two typical courses which were based on this book. A third year Bachelor course (for ambitious students) can be based on parts I and II, i.e. on maximization and applications in discrete and continuous time under certainty. Such a course typically took 14 lectures of 90 minutes each plus the same number of tutorials. It is also possible to present the material also in 14 lectures of 90 minutes each plus only 7 tutorials. Presenting the material without tutorials requires a lot of energy from the students to go through the problem sets on their own. One can, however, discuss some selected problem sets during lectures.

The other typical course which was based on this book is a first-year PhD course. It would review a few chapters of part I and part II (especially the chapters on dynamic programming) and cover in full the stochastic material of part III and part IV. It also requires fourteen 90 minute sessions and exercise classes help even more, given the more complex material. But the same type of arrangements as discussed for the Bachelor course did work as well.

Of course, any other combination is feasible. From my own experience, teaching part I and II in a third year Bachelor course allows teaching of part III and IV at the Master level. Of course, Master courses can be based on any parts of this book, first-year PhD courses can start with part I and II and second-year field courses can use material of part III or IV. This all depends on the ambition of the programme, the intention of the lecturer and the needs of the students.

Apart from being used in classroom, many PhD students and advanced Bachelor or Master students have used various parts of previous versions of this text for studying on their own. Given the detailed step-by-step approach to problems, it turned out that it was very useful for understanding the basic structure of, say, a maximization problem. Once this basic structure was understood, many extensions to more complex problems were obtained - some of which then even found their way into this book.

Part I

Deterministic models in discrete time

This book consists of four parts. In this first part of the book, we will get to know the simplest and therefore maybe the most useful structures to think about changes over time, to think about dynamics. Part I deals with discrete time models under certainty. The first chapter introduces the simplest possible intertemporal problem, a two-period problem. It is solved in a general way and for many functional forms. The methods used are the Lagrangian and simple substitution. Various concepts like the time preference rate and the intertemporal elasticities of substitution are introduced here as well, as they are widely used in the literature and are used frequently throughout this book. For those who want to understand the background of the Lagrangian, a chapter is included that shows the link between Lagrangians and solving by substitution. This will also give us the opportunity to explain the concept of shadow prices as they play an important role e.g. when using Hamiltonians or dynamic programming. The two-period optimal consumption setup will then be put into a decentralized general equilibrium setup. This allows us to understand general equilibrium structures in general while, at the same time, we get to know the standard overlapping generations (OLG) general equilibrium model. This is one of the most widely used dynamic models in Economics. Chapter 2 concludes by reviewing some aspects of difference equations.

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Chapter 2

Two-period models and difference equations

Given that the idea of this book is to start from simple structures and extend them to the more complex ones, this chapter starts with the simplest intertemporal problem, a two-period decision framework. This simple setup already allows us to illustrate the basic dynamic trade-offs. Aggregating over individual behaviour, assuming an overlapping-generations (OLG) structure, and putting individuals in general equilibrium provides an understanding of the issues involved in these steps and in identical steps in more general settings. Some revision of properties of difference equations concludes this chapter.

2.1 Intertemporal utility maximization

2.1.1 The setup

Let there be an individual living for two periods, in the first she is young, in the second she is old. Let her utility function be given by

$$U_t = U\left(c_t^y, c_{t+1}^o\right) \equiv U\left(c_t, c_{t+1}\right), \tag{2.1.1}$$

where consumption when young and old are denoted by c_t^y and c_{t+1}^o or c_t and c_{t+1}, respectively, when no ambiguity arises. The individual earns labour income w_t in both periods. It could also be assumed that she earns labour income only in the first period (e.g. when retiring in the second) or only in the second period (e.g. when going to school in the first). Here, with s_t denoting savings, her budget constraint in the first period is

$$c_t + s_t = w_t \tag{2.1.2}$$

and in the second it reads

$$c_{t+1} = w_{t+1} + \left(1 + r_{t+1}\right) s_t. \tag{2.1.3}$$

Interest paid on savings in the second period are given by r_{t+1}. All quantities are expressed in units of the consumption good (i.e. the price of the consumption good is set equal to one. See ch. 2.2.2 for an extension where the price of the consumption good is p_t.).

This budget constraint says something about the assumptions made on the timing of wage payments and savings. What an individual can spend in period two is principal and interest on her savings s_t of the first period. There are no interest payments in period one. This means that wages are paid and consumption takes place at the end of period 1 and savings are used for some productive purposes (e.g. firms use it in the form of capital for production) in period 2. Therefore, returns r_{t+1} are determined by economic conditions in period 2 and have the index $t + 1$. Timing is illustrated in the following figure.

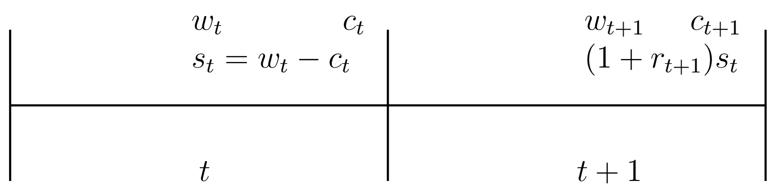


Figure 2.1.1 *Timing in two-period models*

These two budget constraints can be merged into one intertemporal budget constraint by substituting out savings,

$$w_t + (1 + r_{t+1})^{-1} w_{t+1} = c_t + (1 + r_{t+1})^{-1} c_{t+1}. \tag{2.1.4}$$

It should be noted that by not restricting savings to be non-negative in (2.1.2) or by equating the present value of income on the left-hand side with the present value of consumption on the right-hand side in (2.1.4), we assume perfect capital markets: individuals can save and borrow any amount they desire. Equation (2.2.14) provides a condition under which individuals save.

Adding the behavioural assumption that individuals maximize utility, the economic behaviour of an individual is described completely and one can derive her consumption and saving decisions. The problem can be solved by a Lagrange approach or simply by substitution. The latter will be done in ch. 2.2.1 and 3.8.2 in deterministic setups or extensively in ch. 8.1.4 for a stochastic framework. Substitution transforms an optimization problem with a constraint into an unconstrained problem. We will use a Lagrange approach now.

The maximization problem reads $\max_{c_t, c_{t+1}}$(2.1.1) subject to the intertemporal budget constraint (2.1.4). The household's control variables are c_t and c_{t+1}. As they need to be chosen so that they satisfy the budget constraint, they can not be chosen independently of each other. Wages and interest rates are exogenously given to the household. When choosing consumption levels, the reaction of these quantities to the decision of our household is assumed to be zero - simply because the household is too small to have an effect on economy-wide variables.

2.1.2 Solving by the Lagrangian

We will solve this maximization problem by using the Lagrange function. This function will now be presented simply and its structure will be explained in a "recipe sense", which is the most useful one for those interested in quick applications. For those interested in more background, ch. 2.3 will show the formal principles behind the Lagrangian. The Lagrangian for our problem reads

$$\mathcal{L} = U\left(c_t, c_{t+1}\right) + \lambda\left[w_t + \left(1 + r_{t+1}\right)^{-1} w_{t+1} - c_t - \left(1 + r_{t+1}\right)^{-1} c_{t+1}\right], \tag{2.1.5}$$

where λ is a parameter called the Lagrange multiplier. The Lagrangian always consists of two parts. The first part is the objective function, the second part is the product of the Lagrange multiplier and the constraint, expressed as the difference between the right-hand side and the left-hand side of (2.1.4). Technically speaking, it makes no difference whether one subtracts the left-hand side from the right-hand side as here or vice versa - right-hand side minus left-hand side. Reversing the difference would simply change the sign of the Lagrange multiplier but not change the final optimality conditions. Economically, however, one would usually want a positive sign of the multiplier, as we will see in ch. 2.3.

The first-order conditions are

$$\begin{aligned}
\mathcal{L}_{c_t} &= U_{c_t}\left(c_t, c_{t+1}\right) - \lambda = 0, \\
\mathcal{L}_{c_{t+1}} &= U_{c_{t+1}}\left(c_t, c_{t+1}\right) - \lambda\left[1 + r_{t+1}\right]^{-1} = 0, \\
\mathcal{L}_{\lambda} &= w_t + \left(1 + r_{t+1}\right)^{-1} w_{t+1} - c_t - \left(1 + r_{t+1}\right)^{-1} c_{t+1} = 0.
\end{aligned}$$

Clearly, the last first-order condition is identical to the budget constraint. Note that there are three variables to be determined, consumption for both periods and the Lagrange multiplier λ. Having at least three conditions is a necessary, though not sufficient (they might, generally speaking, be linearly dependent) condition for this to be possible.

The first two first-order conditions can be combined to give

$$U_{c_t}\left(c_t, c_{t+1}\right) = \left(1 + r_{t+1}\right) U_{c_{t+1}}\left(c_t, c_{t+1}\right) = \lambda. \tag{2.1.6}$$

Marginal utility from consumption today on the left-hand side must equal marginal utility tomorrow, corrected by the interest rate, on the right-hand side. The economic meaning of this correction can best be understood when looking at a version with nominal budget constraints (see ch. 2.2.2).

We learn from this maximization that the modified first-order condition (2.1.6) gives us a necessary equation that needs to hold when behaving optimally. It links consumption c_t

today to consumption c_{t+1} tomorrow. This equation together with the budget constraint (2.1.4) provides a two-dimensional system: two equations in two unknowns, c_t and c_{t+1}. These equations therefore allow us in principle to compute these endogenous variables as a function of exogenously given wages and interest rates. This would then be the solution to our maximization problem. The next section provides an example where this is indeed the case.

2.2 Examples

2.2.1 Optimal consumption

- The setup

This first example allows us to solve explicitly for consumption levels in both periods. Let preferences of households be represented by

$$U_t = \gamma \ln c_t + (1-\gamma) \ln c_{t+1}. \tag{2.2.1}$$

This utility function is often referred to as Cobb-Douglas or logarithmic utility function. Utility from consumption in each period, instantaneous utility, is given by the logarithm of consumption. Instantaneous utility is sometimes also referred to as felicity function. As $\ln c$ has a positive first and negative second derivative, higher consumption increases instantaneous utility but at a decreasing rate. Marginal utility from consumption is decreasing in (2.2.1). The parameter γ captures the importance of instantaneous utility in the first relative to instantaneous utility in the second. Overall utility U_t is maximized subject to the constraint we know from (2.1.4) above,

$$W_t = c_t + (1 + r_{t+1})^{-1} c_{t+1}, \tag{2.2.2}$$

where we denote the present value of labour income by

$$W_t \equiv w_t + (1 + r_{t+1})^{-1} w_{t+1}. \tag{2.2.3}$$

Again, the consumption good is chosen as numéraire good and its price equals unity. Wages are therefore expressed in units of the consumption good.

- Solving by the Lagrangian

The Lagrangian for this problem reads

$$\mathcal{L} = \gamma \ln c_t + (1-\gamma) \ln c_{t+1} + \lambda \left[W_t - c_t - (1 + r_{t+1})^{-1} c_{t+1} \right].$$

The first-order conditions are

$$\mathcal{L}_{c_t} = \gamma (c_t)^{-1} - \lambda = 0,$$
$$\mathcal{L}_{c_{t+1}} = (1-\gamma)(c_{t+1})^{-1} - \lambda [1 + r_{t+1}]^{-1} = 0,$$

and the budget constraint (2.2.2) following from $\mathcal{L}_\lambda = 0$.

- The solution

Dividing first-order conditions gives $\frac{\gamma}{1-\gamma}\frac{c_{t+1}}{c_t} = 1 + r_{t+1}$ and therefore

$$c_t = \frac{\gamma}{1-\gamma}\left(1 + r_{t+1}\right)^{-1} c_{t+1}.$$

This equation corresponds to our optimality rule (2.1.6) derived above for the more general case. Inserting into the budget constraint (2.2.2) yields

$$W_t = \left(\frac{\gamma}{1-\gamma} + 1\right)\left(1 + r_{t+1}\right)^{-1} c_{t+1} = \frac{1}{1-\gamma}\left(1 + r_{t+1}\right)^{-1} c_{t+1}$$

which is equivalent to

$$c_{t+1} = (1-\gamma)\left(1 + r_{t+1}\right) W_t. \tag{2.2.4}$$

It follows that

$$c_t = \gamma W_t. \tag{2.2.5}$$

Apparently, optimal consumption decisions imply that consumption when young is given by a share γ of the present value W_t of life-time income at time t of the individual under consideration. Consumption when old is given by the remaining share $1-\gamma$ plus interest payments, $c_{t+1} = \left(1 + r_{t+1}\right)(1-\gamma) W_t$. Equations (2.2.4) and (2.2.5) are the solution to our maximization problem. These expressions are sometimes called "closed-form" solutions. A (closed-form) solution expresses the endogenous variable, consumption in our case, as a function only of exogenous variables. Closed-form solution is a different word for closed-loop solution. For further discussion see ch. 5.6.2.

Assuming preferences as in (2.2.1) makes modelling sometimes easier than with more complex utility functions. A drawback here is that the share of lifetime income consumed in the first period and therefore the savings decision is independent of the interest rate, which appears implausible. A way out is given by the CES utility function (see below at (2.2.10)) which also allows for closed-form solutions for consumption (for an example in a stochastic setup, see exercise 6 in ch. 8.1.4). More generally speaking, such a simplification should be justified if some aspect of an economy that is fairly independent of savings is the focus of some analysis.

- Solving by substitution

Let us now consider this example and see how this maximization problem could have been solved without using the Lagrangian. The principle is simply to transform an optimization problem with constraints into an optimization problem without constraints. This is most simply done in our example by replacing the consumption levels in the objective function (2.2.1) by the consumption levels from the constraints (2.1.2) and (2.1.3). The unconstrained maximization problem then reads $\max U_t$ by choosing s_t, where

$$U_t = \gamma \ln\left(w_t - s_t\right) + (1-\gamma)\ln\left(w_{t+1} + \left(1 + r_{t+1}\right) s_t\right).$$

This objective function shows the trade-off faced by anyone who wants to save nicely. High savings reduce consumption today but increase consumption tomorrow.

The first-order condition for savings yields after some simple algebra (see ex. 2 and the solutions manual for details)

$$s_t = w_t - \gamma W_t,$$

where W_t is life-time income as defined after (2.2.2). To see that this is consistent with the solution by Lagrangian, compute first-period consumption and find $c_t = w_t - s_t = \gamma W_t$ - which is the solution in (2.2.5).

What have we learned from using this substitution method? We see that we do not need "sophisticated" tools like the Lagrangian as we can solve a "normal" unconstrained problem - the type of problem we might be more familiar with from static maximization setups. But the steps to obtain the final solution appear somewhat more "curvy" and less elegant. It therefore appears worthwhile to become more familiar with the Lagrangian.

2.2.2 Optimal consumption with prices

Consider now again the utility function (2.1.1) and maximize it subject to the constraints $p_t c_t + s_t = w_t$ and $p_{t+1} c_{t+1} = w_{t+1} + (1 + r_{t+1}) s_t$. The difference to the introductory example in ch. 2.1 consists in the introduction of an explicit price p_t for the consumption good. The first-period budget constraint now equates nominal expenditure for consumption ($p_t c_t$ is measured in, say, Euro, Dollar or Yen) plus nominal savings to nominal wage income. The second period constraint equally equates nominal quantities. What does an optimal consumption rule as in (2.1.6) now look like?

The Lagrangian is, now using an intertemporal budget constraint with prices,

$$\mathcal{L} = U(c_t, c_{t+1}) + \lambda \left[W_t - p_t c_t - (1 + r_{t+1})^{-1} p_{t+1} c_{t+1} \right].$$

The first-order conditions for c_t and c_{t+1} are

$$\mathcal{L}_{c_t} = U_{c_t}(c_t, c_{t+1}) - \lambda p_t = 0,$$
$$\mathcal{L}_{c_{t+1}} = U_{c_{t+1}}(c_t, c_{t+1}) - \lambda (1 + r_{t+1})^{-1} p_{t+1} = 0$$

and the intertemporal budget constraint. Combining them gives

$$\frac{U_{c_t}(c_t, c_{t+1})}{p_t} = \frac{U_{c_{t+1}}(c_t, c_{t+1})}{p_{t+1}[1 + r_{t+1}]^{-1}} \Leftrightarrow \frac{U_{c_t}(c_t, c_{t+1})}{U_{c_{t+1}}(c_t, c_{t+1})} = \frac{p_t}{p_{t+1}[1 + r_{t+1}]^{-1}}. \quad (2.2.6)$$

This equation says that marginal utility of consumption today relative to marginal utility of consumption tomorrow equals the relative price of consumption today and tomorrow. This optimality rule is identical for a static 2-good decision problem where optimality requires that the ratio of marginal utilities equals the relative price. The relative price here is expressed in a present value sense as we compare marginal utilities at two points in time. The price p_t is therefore divided by the price tomorrow, discounted by the next period's interest rate, $p_{t+1}\left[1+r_{t+1}\right]^{-1}$.

In contrast to what sometimes seems common practice, we will *not* call (2.2.6) a solution to the maximization problem. This expression (frequently referred to as the Euler equation) is simply an expression resulting from first-order conditions. Strictly speaking, (2.2.6) is only a necessary condition for optimal behaviour - and not more. As defined above, a solution to a maximization problem is a closed-form expression as for example in (2.2.4) and (2.2.5). It gives information on levels - and not just on changes as in (2.2.6). Being aware of this important difference, in what follows, the term "solving a maximization problem" will nevertheless cover both analyses. Those which stop at the Euler equation and those which go all the way towards obtaining a closed-form solution.

2.2.3 Some useful definitions with applications

In order to be able to discuss results in subsequent sections easily, we review some definitions here that will be used frequently in later parts of this book. We are mainly interested in the intertemporal elasticity of substitution and the time preference rate. While a lot of this material can be found in micro textbooks, the notation used in these books differs of course from the one used here. As this book is also intended to be as self-contained as possible, this short review can serve as a reference for subsequent explanations. We start with the

- Marginal rate of substitution (MRS)

Let there be a consumption bundle $(c_1, c_2, ..., c_n)$. Let utility be given by $u(c_1, c_2, ..., c_n)$ which we abbreviate to $u(.)$. The MRS between good i and good j is then defined by

$$MRS_{ij}(.) \equiv \frac{\partial u(.)/\partial c_i}{\partial u(.)/\partial c_j}. \tag{2.2.7}$$

It gives the *increase* of consumption of good j that is required to keep the utility level at $u(c_1, c_2, ..., c_n)$ when the amount of i is *decreased* marginally. By this definition, this amount is positive if both goods are normal goods - i.e. if both partial derivatives in (2.2.7) are positive. Note that definitions used in the literature can differ from this one. Some replace '*decreased*' by 'increased' (or - which has the same effect - replace '*increase*' by 'decrease') and thereby obtain a different sign.

Why this is so can easily be shown: Consider the total differential of $u(c_1, c_2, ..., c_n)$, keeping all consumption levels apart from c_i and c_j fix. This yields

$$du(c_1, c_2, ..., c_n) = \frac{\partial u(.)}{\partial c_i}dc_i + \frac{\partial u(.)}{\partial c_j}dc_j.$$

The overall utility level $u(c_1, c_2, ..., c_n)$ does not change if

$$du(.) = 0 \Leftrightarrow \frac{dc_j}{dc_i} = -\frac{\partial u(.)}{\partial c_i}/\frac{\partial u(.)}{\partial c_j} \equiv -MRS_{ij}(.).$$

- Equivalent terms

As a reminder, the equivalent term to the MRS in production theory is the marginal rate of technical substitution $MRTS_{ij}(.) = \frac{\partial f(.)/\partial x_i}{\partial f(.)/\partial x_j}$ where the utility function was replaced by a production function and consumption c_k was replaced by factor inputs x_k.

On a more economy-wide level, there is the marginal rate of transformation $MRT_{ij}(.) = \frac{\partial G(.)/\partial y_i}{\partial G(.)/\partial y_j}$ where the utility function has now been replaced by a transformation function G (maybe better known as production possibility curve) and the y_k are output of good k. The marginal rate of transformation gives the increase in output of good j when output of good i is marginally decreased.

- (Intertemporal) elasticity of substitution

Though our main interest is a measure of *intertemporal* substitutability, we first define the elasticity of substitution in general. As with the marginal rate of substitution, the definition implies a certain sign of the elasticity. In order to obtain a positive sign (with normal goods), we define the elasticity of substitution as the *increase* in relative consumption c_i/c_j when the relative price p_i/p_j *decreases* (which is equivalent to an increase of p_j/p_i). Formally, we obtain for the case of two consumption goods $\epsilon_{ij} \equiv \frac{d(c_i/c_j)}{d(p_j/p_i)} \frac{p_j/p_i}{c_i/c_j}$.

This definition can be expressed alternatively (see ex. 6 for details) in a way which is more useful for the examples below. We express the elasticity of substitution by the derivative of the log of relative consumption c_i/c_j with respect to the log of the marginal rate of substitution between j and i,

$$\epsilon_{ij} \equiv \frac{d \ln (c_i/c_j)}{d \ln MRS_{ji}}. \tag{2.2.8}$$

Inserting the marginal rate of substitution MRS_{ji} from (2.2.7), i.e. exchanging i and j in (2.2.7), gives

$$\epsilon_{ij} = \frac{d \ln (c_i/c_j)}{d \ln \left(u_{c_j}/u_{c_i}\right)} = \frac{u_{c_j}/u_{c_i}}{c_i/c_j} \frac{d (c_i/c_j)}{d \left(u_{c_j}/u_{c_i}\right)}.$$

The advantage of an elasticity when compared to a normal derivative, such as the MRS, is that an elasticity is measureless. It is expressed in percentage changes. (This can be best seen in the following example and in ex. 6 where the derivative is multiplied by $\frac{p_j/p_i}{c_i/c_j}$.) It can both be applied to static utility or production functions or to intertemporal utility functions.

The intertemporal elasticity of substitution for a utility function $u(c_t, c_{t+1})$ is then simply the elasticity of substitution of consumption at two points in time,

$$\epsilon_{t,t+1} \equiv \frac{u_{c_t}/u_{c_{t+1}}}{c_{t+1}/c_t} \frac{d (c_{t+1}/c_t)}{d \left(u_{c_t}/u_{c_{t+1}}\right)}. \tag{2.2.9}$$

Here as well, in order to obtain a positive sign, the subscripts in the denominator have a different ordering from the one in the numerator.

- The intertemporal elasticity of substitution for logarithmic and CES utility functions

For the logarithmic utility function $U_t = \gamma \ln c_t + (1-\gamma) \ln c_{t+1}$ from (2.2.1), we obtain an intertemporal elasticity of substitution of one,

$$\epsilon_{t,t+1} = \frac{\frac{\gamma}{c_t} / \frac{1-\gamma}{c_{t+1}}}{c_{t+1}/c_t} \frac{d\left(c_{t+1}/c_t\right)}{d\left(\frac{\gamma}{c_t}/\frac{1-\gamma}{c_{t+1}}\right)} = 1,$$

where the last step used

$$\frac{d\left(c_{t+1}/c_t\right)}{d\left(\frac{\gamma}{c_t}/\frac{1-\gamma}{c_{t+1}}\right)} = \frac{1-\gamma}{\gamma} \frac{d\left(c_{t+1}/c_t\right)}{d\left(c_{t+1}/c_t\right)} = \frac{1-\gamma}{\gamma}.$$

It is probably worth noting at this point that not all textbooks would agree on the result of "plus one". Following some other definitions, a result of minus one would be obtained. Keeping in mind that the sign is just a convention, depending on "increase" or "decrease" in the definition, this should not lead to confusions.

When we consider a utility function where instantaneous utility is not logarithmic but of CES type

$$U_t = \gamma c_t^{1-\sigma} + (1-\gamma) c_{t+1}^{1-\sigma}, \tag{2.2.10}$$

the intertemporal elasticity of substitution becomes

$$\epsilon_{t,t+1} \equiv \frac{\gamma\left[1-\sigma\right] c_t^{-\sigma} / (1-\gamma)(1-\sigma) c_{t+1}^{-\sigma}}{c_{t+1}/c_t} \frac{d\left(c_{t+1}/c_t\right)}{d\left(\gamma\left[1-\sigma\right] c_t^{-\sigma} / (1-\gamma)(1-\sigma) c_{t+1}^{-\sigma}\right)}. \tag{2.2.11}$$

Defining $x \equiv \left(c_{t+1}/c_t\right)^{\sigma}$, we obtain

$$\begin{aligned} \frac{d\left(c_{t+1}/c_t\right)}{d\left(\gamma\left[1-\sigma\right] c_t^{-\sigma} / (1-\gamma)(1-\sigma) c_{t+1}^{-\sigma}\right)} &= \frac{1-\gamma}{\gamma} \frac{d\left(c_{t+1}/c_t\right)}{d\left(c_t^{-\sigma}/c_{t+1}^{-\sigma}\right)} = \frac{1-\gamma}{\gamma} \frac{dx^{1/\sigma}}{dx} \\ &= \frac{1-\gamma}{\gamma} \frac{1}{\sigma} x^{\frac{1}{\sigma}-1}. \end{aligned}$$

Inserting this into (2.2.11) and cancelling terms, the elasticity of substitution turns out to be

$$\epsilon_{t,t+1} \equiv \frac{c_t^{-\sigma}/c_{t+1}^{-\sigma}}{c_{t+1}/c_t} \frac{1}{\sigma} \left(\frac{c_{t+1}}{c_t}\right)^{1-\sigma} = \frac{1}{\sigma}.$$

This is where the CES utility function (2.2.10) has its name from: The intertemporal elasticity (E) of substitution (S) is constant (C).

- The time preference rate

Intuitively, the time preference rate is the rate at which future instantaneous utilities are discounted. To illustrate, imagine a discounted income stream

$$x_0 + \frac{1}{1+r}x_1 + \left(\frac{1}{1+r}\right)^2 x_2 + ...$$

where discounting takes place at the interest rate r. Replacing income x_t by instantaneous utility and the interest rate by ρ, ρ would be the time preference rate. Formally, the time preference rate is the marginal rate of substitution of instantaneous utilities (not of consumption levels) minus one,

$$TPR \equiv MRS_{t,t+1} - 1.$$

As an example, consider the following standard utility function which we will use very often in later chapters,

$$U_0 = \Sigma_{t=0}^{\infty}\beta^t u(c_t), \qquad \beta \equiv \frac{1}{1+\rho}, \qquad \rho > 0. \tag{2.2.12}$$

Let ρ be a positive parameter and β the implied discount factor, capturing the idea of impatience: By multiplying instantaneous utility functions $u(c_t)$ by β^t, future utility is valued less than present utility. This utility function generalizes (2.2.1) in two ways: First and most importantly, there is a much longer planning horizon than just two periods. In fact, the individual's overall utility U_0 stems from the sum of discounted instantaneous utility levels $u(c_t)$ over periods 0, 1, 2, ... up to infinity. The idea behind this objective function is not that individuals live forever but that individuals care about the well-being of subsequent generations. Second, the instantaneous utility function $u(c_t)$ is not logarithmic as in (2.2.1) but of a more general nature where one would usually assume positive first and negative second derivatives, $u' > 0$, $u'' < 0$.

The marginal rate of substitution is then

$$MRS_{t,t+1}(.) = \frac{\partial U_0(.)/\partial u(c_t)}{\partial U_0(.)/\partial u(c_{t+1})} = \frac{(1/(1+\rho))^t}{(1/(1+\rho))^{t+1}} = 1+\rho.$$

The time preference rate is therefore given by ρ.

Now take for example the utility function (2.2.1). Computing the MRS minus one, we have

$$\rho = \frac{\gamma}{1-\gamma} - 1 = \frac{2\gamma-1}{1-\gamma}. \tag{2.2.13}$$

The time preference rate is positive if $\gamma > 0.5$. This makes sense for (2.2.1) as one should expect that future utility is valued less than present utility.

As a side note, all intertemporal utility functions in this book will use exponential discounting as in (2.2.12). This is clearly a special case. Models with non-exponential or hyperbolic discounting imply fundamentally different dynamic behaviour and time inconsistencies. See "further reading" for some references.

- Does consumption increase over time?

This definition of the time preference rate allows us to provide a precise answer to the question whether consumption increases over time. We simply compute the condition under which $c_{t+1} > c_t$ by using (2.2.4) and (2.2.5),

$$c_{t+1} > c_t \Leftrightarrow (1-\gamma)(1+r_{t+1}) W_t > \gamma W_t \Leftrightarrow$$
$$1 + r_{t+1} > \frac{\gamma}{1-\gamma} \Leftrightarrow r_{t+1} > \frac{\gamma - 1 + \gamma}{1-\gamma} \Leftrightarrow r_{t+1} > \rho.$$

Consumption increases if the interest rate is higher than the time preference rate. The time preference rate of the individual (being represented by γ) determines how to split the present value W_t of total income into current and future use. If the interest rate is sufficiently high to overcompensate impatience, i.e. if $(1-\gamma)(1+r) > \gamma$ in the first line, consumption rises.

Note that even though we computed the condition for rising consumption for our special utility function (2.2.1), the result that consumption increases when the interest rate exceeds the time preference rate holds for more general utility functions as well. We will get to know various examples for this in subsequent chapters.

- Under what conditions are savings positive?

Savings are from the budget constraint (2.1.2) and the optimal consumption result (2.2.5) given by

$$s_t = w_t - c_t = w_t - \gamma \left[w_t + \frac{1}{1+r_{t+1}} w_{t+1} \right] = w \left[1 - \gamma - \frac{\gamma}{1+r_{t+1}} \right]$$

where the last equality assumed an invariant wage level, $w_t = w_{t+1} \equiv w$. Savings are positive if and only if

$$s_t > 0 \Leftrightarrow 1 - \gamma > \frac{\gamma}{1+r_{t+1}} \Leftrightarrow 1 + r_{t+1} > \frac{\gamma}{1-\gamma} \Leftrightarrow$$
$$r_{t+1} > \frac{2\gamma - 1}{1-\gamma} \Leftrightarrow r_{t+1} > \rho \qquad (2.2.14)$$

This means that savings are positive if interest rate is larger than time preference rate. Clearly, this result does not necessarily hold for $w_{t+1} > w_t$.

2.3 The idea behind the Lagrangian

So far, we simply used the Lagrange function without asking where it comes from. This chapter will offer a derivation of the Lagrange function and also an economic interpretation of the Lagrange multiplier. In maximization problems employing a utility function, the Lagrange multiplier can be understood as a price measured in utility units. It is often called a shadow price.

2.3.1 Where the Lagrangian comes from I

- The maximization problem

Let us consider a maximization problem with some objective function and a constraint,

$$\max_{x_1,x_2} F(x_1,x_2) \text{ subject to } g(x_1,x_2)=b. \tag{2.3.1}$$

The constraint can be looked at as an implicit function, i.e. describing x_2 as a function of x_1, i.e. $x_2 = h(x_1)$. Using the representation $x_2 = h(x_1)$ of the constraint, the maximization problem can be written as

$$\max_{x_1} F(x_1, h(x_1)). \tag{2.3.2}$$

- The derivatives of implicit functions

As we will use implicit functions and their derivatives here and in later chapters, we briefly illustrate the underlying idea and show how to compute their derivatives. Consider a function $f(x_1, x_2, ..., x_n) = 0$. The implicit function theorem says - stated simply - that this function $f(x_1, x_2, ..., x_n) = 0$ implicitly defines (under suitable assumptions concerning the properties of $f(.)$ - see exercise 7) a functional relationship of the type $x_2 = h(x_1, x_3, x_4, ..., x_n)$. We often work with these implicit functions in Economics and we are also often interested in the derivative of x_2 with respect to, say, x_1.

In order to obtain an expression for this derivative, consider the total differential of $f(x_1, x_2, ..., x_n) = 0$,

$$df(.) = \frac{\partial f(.)}{\partial x_1}dx_1 + \frac{\partial f(.)}{\partial x_2}dx_2 + ... + \frac{\partial f(.)}{\partial x_n}dx_n = 0.$$

When we keep x_3 to x_n constant, we can solve this to get

$$\frac{dx_2}{dx_1} = -\frac{\partial f(.)/\partial x_1}{\partial f(.)/\partial x_2}. \tag{2.3.3}$$

We have thereby obtained an expression for the derivative dx_2/dx_1 without knowing the functional form of the implicit function $h(x_1, x_3, x_4, ..., x_n)$.

For illustration purposes, consider the following figure.

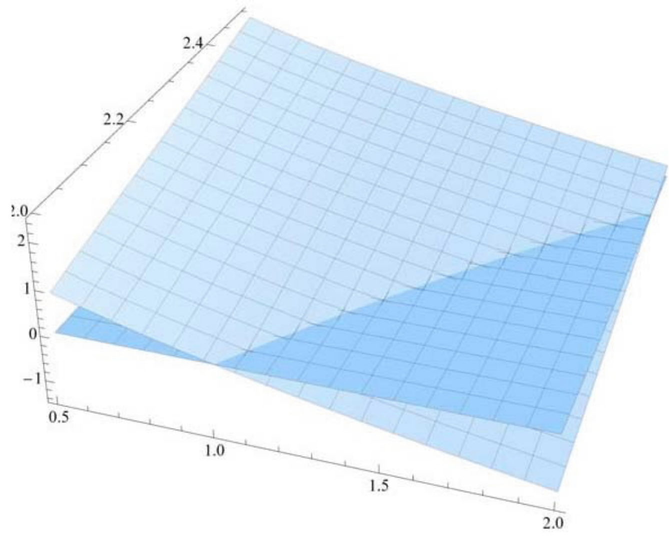


Figure 2.3.1 *The implicit function visible at $z = 0$*

The horizontal axes plot x_1 and, to the back, x_2. The vertical axis plots z. The increasing surface depicts the graph of the function $z = g(x_1, x_2) - b$. When this surface crosses the horizontal plane at $z = 0$, a curve is created which contains all the points where $z = 0$. Looking at this curve illustrates that the function $z = 0 \Leftrightarrow g(x_1, x_2) = b$ implicitly defines a function $x_2 = h(x_1)$. See exercise 7 for an explicit analytical derivation of such an implicit function. The derivative dx_2/dx_1 is then simply the slope of this curve. The analytical expression for this is - using (2.3.3) - $dx_2/dx_1 = -(\partial g(.)/\partial x_1)/(\partial g(.)/\partial x_2)$.

- First-order conditions of the maximization problem

The maximization problem we obtained in (2.3.2) is an example for the substitution method: The budget constraint was solved for one control variable and inserted into the objective function. The resulting maximization problem is one without constraint. The problem in (2.3.2) now has a standard first-order condition,

$$\frac{dF}{dx_1} = \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2}\frac{dh}{dx_1} = 0. \tag{2.3.4}$$

Taking into consideration that from the implicit function theorem applied to the constraint,

$$\frac{dh}{dx_1} = \frac{dx_2}{dx_1} = -\frac{\partial g(x_1, x_2)/\partial x_1}{\partial g(x_1, x_2)/\partial x_2}, \tag{2.3.5}$$

the optimality condition (2.3.4) can be written as $\frac{\partial F}{\partial x_1} - \frac{\partial F/\partial x_2}{\partial g/\partial x_2}\frac{\partial g(x_1,x_2)}{\partial x_1} = 0$. Now define the Lagrange multiplier $\lambda \equiv \frac{\partial F/\partial x_2}{\partial g/\partial x_2}$ and obtain

$$\frac{\partial F}{\partial x_1} - \lambda\frac{\partial g(x_1, x_2)}{\partial x_1} = 0. \tag{2.3.6}$$

As can be easily seen, this is the first-order condition of the Lagrangian

$$\mathcal{L} = F(x_1, x_2) + \lambda [b - g(x_1, x_2)] \tag{2.3.7}$$

with respect to x_1.

Now imagine we want to undertake the same steps for x_2, i.e. we start from the original problem (2.3.1) but substitute out x_1. We would then obtain an unconstrained problem as in (2.3.2) only that we maximize with respect to x_2. Continuing as we just did for x_1 would yield the second first-order condition

$$\frac{\partial F}{\partial x_2} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_2} = 0.$$

We have thereby shown where the Lagrangian comes from: Whether one defines a Lagrangian as in (2.3.7) and computes the first-order condition or one computes the first-order condition from the unconstrained problem as in (2.3.4) and then uses the implicit function theorem and defines a Lagrange multiplier, one always ends up at (2.3.6). The Lagrangian-route is obviously faster.

2.3.2 Shadow prices

- The idea

We can now also give an interpretation of the meaning of the multipliers λ. Starting from the definition of λ in (2.3.6), we can rewrite it according to

$$\lambda \equiv \frac{\partial F / \partial x_2}{\partial g / \partial x_2} = \frac{\partial F}{\partial g} = \frac{\partial F}{\partial b}.$$

One can understand that the first equality can "cancel" the term ∂x_2 by looking at the definition of a (partial) derivative: $\frac{\partial f(x_1,...,x_n)}{\partial x_i} = \frac{\lim_{\triangle x_i \to 0} f(x_1,...,x_i + \triangle x_i,...,x_n) - f(x_1,...,x_n)}{\lim_{\triangle x_i \to 0} \triangle x_i}$. The second equality uses the equality of g and b from the constraint in (2.3.1). From these transformations, we see that λ equals the change in F as a function of b. It is now easy to come up with examples for F or b: How much does F increase (e.g. your utility) when your constraint b (your bank account) is relaxed? How much does the social welfare function change when the economy has more capital? How much do profits of firms change when the firm has more workers? This λ is called shadow price and expresses the value of b in units of F.

- A derivation

A more rigorous derivation is as follows (cf. Intriligator, 1971, ch. 3.3). Compute the derivative of the maximized Lagrangian with respect to b,

$$\begin{aligned}\frac{\partial \mathcal{L}(x_1^*(b), x_2^*(b))}{\partial b} &= \frac{\partial}{\partial b}(F(x_1^*(b), x_2^*(b)) + \lambda(b)\left[b - g\left(x_1^*(b), x_2^*(b)\right)\right] \\ &= F_{x_1^*}\frac{\partial x_1^*}{\partial b} + F_{x_2^*}\frac{\partial x_2^*}{\partial b} + \lambda'(b)\left[b - g(.)\right] + \lambda(b)\left[1 - g_{x_1^*}\frac{\partial x_1^*}{\partial b} - g_{x_2^*}\frac{\partial x_2^*}{\partial b}\right] \\ &= \lambda(b)\end{aligned}$$

The last equality results from first-order conditions and the fact that the budget constraint holds.

As $\mathcal{L}(x_1^*, x_2^*) = F(x_1^*, x_2^*)$ due to the budget constraint holding with equality,

$$\lambda(b) = \frac{\partial \mathcal{L}(x_1^*, x_2^*)}{\partial b} = \frac{\partial F(x_1^*, x_2^*)}{\partial b}$$

- An example

The Lagrange multiplier λ is frequently referred to as shadow price. As we have seen, its unit depends on the unit of the objective function F. One can think of price in the sense of a price in a currency, for example in Euro, only if the objective function is some nominal expression like profits or GDP. Otherwise it is a price expressed for example in utility terms. This can explicitly be seen in the following example. Consider a central planner that maximizes social welfare $u(x_1, x_2)$ subject to technological and resource constraints,

$$\max u(x_1, x_2)$$

subject to

$$\begin{aligned} x_1 &= f(K_1, L_1), \qquad & x_2 &= g(K_2, L_2), \\ K_1 + K_2 &= K, & L_1 + L_2 &= L. \end{aligned}$$

Technologies in sectors 1 and 2 are given by $f(.)$ and $g(.)$ and factors of production are capital K and labour L. Using as multipliers p_1, p_2, w^K and w^L, the Lagrangian reads

$$\begin{aligned}\mathcal{L} = {} & u(x_1, x_2) + p_1\left[f(K_1, L_1) - x_1\right] + p_2\left[g(K_2, L_2) - x_2\right] \\ & + w^K\left[K - K_1 - K_2\right] + w^L\left[L - L_1 - L_2\right]\end{aligned} \tag{2.3.8}$$

and first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial u}{\partial x_1} - p_1 = 0, \qquad & \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial u}{\partial x_2} - p_2 = 0, \\ \frac{\partial \mathcal{L}}{\partial K_1} &= p_1\frac{\partial f}{\partial K_1} - w^K = 0, & \frac{\partial \mathcal{L}}{\partial K_2} &= p_2\frac{\partial g}{\partial K_2} - w^K = 0, \\ \frac{\partial \mathcal{L}}{\partial L_1} &= p_1\frac{\partial f}{\partial L_1} - w^L = 0, & \frac{\partial \mathcal{L}}{\partial L_2} &= p_2\frac{\partial g}{\partial L_2} - w^L = 0. \end{aligned} \tag{2.3.9}$$

Here we see that the first multiplier p_1 is not a price expressed in some currency but the derivative of the utility function with respect to good 1, i.e. marginal utility. By contrast, if we looked at the multiplier w^K only in the third first-order condition, $w^K = p_1 \partial f / \partial K_1$, we would then conclude that it is a price. Then inserting the first first-order condition, $\partial u / \partial x_1 = p_1$, and using the constraint $x_1 = f(K_1, L_1)$ shows however that it really stands for the increase in utility when the capital stock used in production of good 1 rises,

$$w^K = p_1 \frac{\partial f}{\partial K_1} = \frac{\partial u}{\partial x_1} \frac{\partial f}{\partial K_1} = \frac{\partial u}{\partial K_1}.$$

Hence w^K and all other multipliers are prices in utility units.

It is now also easy to see that all shadow prices are prices expressed in some currency if the objective function is not utility but, for example GDP. Such a maximization problem could read $\max p_1 x_1 + p_2 x_2$ subject to the constraints as above. Finally, returning to the discussion after (2.1.5), the first-order conditions show that the sign of the Lagrange multiplier should be positive from an economic perspective. If p_1 in (2.3.9) is to capture the value attached to x_1 in utility units and x_1 is a normal good (utility increases in x_1, i.e. $\partial u / \partial x_1 > 0$), the shadow price should be positive. If we had represented the constraint in the Lagrangian (2.3.8) as $x_1 - f(K_1, L_1)$ rather than right-hand side minus left-hand side, the first-order condition would read $\partial u / \partial x_1 + p_1 = 0$ and the Lagrange multiplier would have been negative. If we want to associate the Lagrange multiplier to the shadow price, the constraints in the Lagrange function should be represented such that the Lagrange multiplier is positive.

2.4 An overlapping generations model

We will now analyze many households jointly and see how their consumption and saving behaviour affects the evolution of the economy as a whole. We will get to know the Euler theorem and how it is used to sum factor incomes to yield GDP. We will also understand how the interest rate in the household's budget constraint is related to marginal productivity of capital and the depreciation rate. All this jointly yields time paths of aggregate consumption, the capital stock and GDP. We will assume an overlapping-generations structure (OLG).

A model in general equilibrium is described by fundamentals of the model and market and behavioural assumptions. Fundamentals are technologies of firms, preferences of households and factor endowments. Adding clearing conditions for markets and behavioural assumptions for agents completes the description of the model.

2.4.1 Technologies

- The firms

Let there be many firms who employ capital K_t and labour L to produce output Y_t according to the technology

$$Y_t = Y(K_t, L). \tag{2.4.1}$$

Assume production of the final good $Y(.)$ is characterized by constant returns to scale. We choose Y_t as our numéraire good and normalize its price to unity, $p_t = 1$. While this is not necessary and we could keep the price p_t all through the model, we would see that all prices, like for example factor rewards, would be expressed relative to the price p_t. Hence, as a shortcut, we set $p_t = 1$. We now, however, need to keep in mind that all prices are henceforth expressed in units of this final good. With this normalization, profits are given by $\pi_t = Y_t - w_t^K K_t - w_t^L L$. Letting firms act under perfect competition, the first-order conditions from profit maximization reflect the fact that the firm takes all prices as parametric and set marginal productivities equal to real factor rewards,

$$\frac{\partial Y_t}{\partial K_t} = w_t^K, \quad \frac{\partial Y_t}{\partial L} = w_t^L. \tag{2.4.2}$$

In each period they equate t, the marginal productivity of capital, to the factor price w_t^K for capital and the marginal productivity of labour to labour's factor reward w_t^L.

- Euler's theorem

Euler's theorem shows that for a linear-homogeneous function $f(x_1, x_2, ..., x_n)$ the sum of partial derivatives times the variables with respect to which the derivative was computed equals the original function $f(.)$,

$$f(x_1, x_2, ..., x_n) = \frac{\partial f(.)}{\partial x_1} x_1 + \frac{\partial f(.)}{\partial x_2} x_2 + ... + \frac{\partial f(.)}{\partial x_n} x_n. \tag{2.4.3}$$

Provided that the technology used by firms to produce Y_t has constant returns to scale, we obtain from this theorem that

$$Y_t = \frac{\partial Y_t}{\partial K_t} K_t + \frac{\partial Y_t}{\partial L} L. \tag{2.4.4}$$

Using the optimality conditions (2.4.2) of the firm for the applied version of Euler's theorem (2.4.4) yields

$$Y_t = w_t^K K_t + w_t^L L. \tag{2.4.5}$$

Total output in this economy, Y_t, is identical to total factor income. This result is usually given the economic interpretation that under perfect competition all revenue in firms is used to pay factors of production. As a consequence, profits π_t of firms are zero.

2.4.2 Households

- Individual households

Households live again for two periods. The utility function is therefore as in (2.2.1) and given by

$$U_t = \gamma \ln c_t^y + (1-\gamma) \ln c_{t+1}^o. \tag{2.4.6}$$

It is maximized subject to the intertemporal budget constraint

$$w_t^L = c_t^y + (1+r_{t+1})^{-1} c_{t+1}^o.$$

This constraint differs slightly from (2.2.2) in that people work only in the first period and retire in the second. Hence, there is labour income only in the first period on the left-hand side. Savings from the first period are used to finance consumption in the second period.

Given that the present value of lifetime wage income is w_t^L, we can conclude from (2.2.4) and (2.2.5) that individual consumption expenditure and savings are given by

$$c_t^y = \gamma w_t^L, \quad c_{t+1}^o = (1-\gamma)(1+r_{t+1}) w_t^L, \tag{2.4.7}$$

$$s_t = w_t^L - c_t^y = (1-\gamma) w_t^L. \tag{2.4.8}$$

- Aggregation

We assume that in each period L individuals are born and die. Hence, the number of young and the number of old is L as well. As all individuals within a generation are identical, aggregate consumption within one generation is simply the number of, say, young times individual consumption. Aggregate consumption in t is therefore given by $C_t = Lc_t^y + Lc_t^o$. Using the expressions for individual consumption from (2.4.7) and noting the index t (and not $t+1$) for the old yields

$$C_t = Lc_t^y + Lc_t^o = \left(\gamma w_t^L + (1-\gamma)(1+r_t) w_{t-1}^L\right) L.$$

2.4.3 Goods market equilibrium and accumulation identity

The goods market equilibrium requires that supply equals demand, $Y_t = C_t + I_t$, where demand is given by consumption plus gross investment. Next period's capital stock is - by an accounting identity - given by $K_{t+1} = I_t + (1-\delta) K_t$. Net investment, amounting to the change in the capital stock, $K_{t+1} - K_t$, is given by gross investment I_t minus depreciation δK_t, where δ is the depreciation rate, $K_{t+1} - K_t = I_t - \delta K_t$. Replacing gross investment by the goods market equilibrium, we obtain the resource constraint

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t. \tag{2.4.9}$$

For our OLG setup, it is useful to rewrite this constraint slightly ,

$$Y_t + (1-\delta) K_t = C_t + K_{t+1}. \tag{2.4.10}$$

In this formulation, it reflects a "broader" goods market equilibrium where the left-hand side shows supply as current production *plus* capital held by the old. The old sell capital as it is of no use for them, given that they will not be able to consume anything in the next period. Demand for the aggregate good is given by aggregate consumption (i.e. consumption of the young plus consumption of the old) plus the capital stock to be held next period by the currently young.

2.4.4 The reduced form

For the first time in this book we have come to the point where we need to find what will be called a "reduced form". Once all maximization problems are solved and all constraints and market equilibria are taken into account, the objective consists of understanding properties of the model, i.e. understanding its predictions. This is usually done by first simplifying the structure of the system of equations coming out of the model as much as possible. In the end, after inserting and reinserting, a system of n equations in n unknowns results. The system where n is the smallest possible is what will be called the reduced form.

Ideally, there is only one equation left and this equation gives an explicit solution of the endogenous variable. In static models, an example would be $L_X = \alpha L$, i.e. employment in sector X is given by a utility parameter α times the total exogenous labour supply L. This would be an explicit solution. If we are left with just one equation but we obtain on an implicit solution, we would obtain something like $f(L_X, \alpha, L) = 0$.

- Deriving the reduced form

We now derive, given the results we have obtained so far, how large the capital stock in the next period is. Splitting aggregate consumption into consumption of the young and consumption of the old and using the output-factor reward identity (2.4.5) for the resource constraint in the OLG case (2.4.10), we obtain

$$w_t^K K_t + w_t^L L + (1-\delta) K_t = C_t^y + C_t^o + K_{t+1}.$$

Defining the interest rate r_t as the difference between factor rewards w_t^K for capital and the depreciation rate δ,

$$r_t \equiv w_t^K - \delta, \tag{2.4.11}$$

we find

$$r_t K_t + w_t^L L + K_t = C_t^y + C_t^o + K_{t+1}.$$

The interest rate definition (2.4.11) shows the net income of capital owners per unit of capital. They earn the gross factor rewards w_t^K but, at the same time, they experience a loss from depreciation. Net income therefore only amounts to r_t. As the old consume the capital stock plus interest $c_t^o L = (1 + r_t) K_t$, we obtain

$$K_{t+1} = w_t^L L - C_t^y = s_t L. \tag{2.4.12}$$

which is the aggregate version of the savings equation (2.4.8). Hence, we have found that savings s_t of young at t is the capital stock at $t+1$.

Note that equation (2.4.12) is often present on "intuitive" grounds. The old in period t have no reason to save as they will not be able to use their savings in $t+1$. Hence, only the young will save and the capital stock in $t+1$, being made up from savings in the previous period, must be equal to the savings of the young.

- The one-dimensional difference equation

In our simple dynamic model considered here, we obtain the ideal case where we are left with only one equation that gives us the solution for one variable, the capital stock. Inserting the individual savings equation (2.4.8) into (2.4.12) gives with the first-order condition (2.4.2) of the firm

$$K_{t+1} = (1-\gamma)\, w_t^L L = (1-\gamma) \frac{\partial Y(K_t, L)}{\partial L} L. \qquad (2.4.13)$$

The first equality shows that a share $1-\gamma$ of labour income turns into capital in the next period. Interestingly, the depreciation rate does not have an impact on the capital stock in period $t+1$. Economically speaking, the depreciation rate affects the wealth of the old but - with logarithmic utility - not the saving of the young.

2.4.5 Properties of the reduced form

Equation (2.4.13) is a non-linear difference equation in K_t. All other quantities in this equation are constant. This equation determines the entire path of capital in this dynamic economy, provided we have an initial condition K_0. We have therefore indeed solved the maximization problem and reduced the general equilibrium model to one single equation. From the path of capital, we can compute all other variables which are of interest for our economic questions.

Whenever we have reduced a model to its reduced form and have obtained one or more difference equations (or differential equations in continuous time), we would like to understand the properties of such a dynamic system. The procedure is in principle always the same: We first ask whether there is some solution where all variables (K_t in our case) are constant. This is then called a steady state analysis. Once we have understood the steady state (if there is one), we want to understand how the economy behaves out of the steady state, i.e. what its transitional dynamics are.

- Steady state

In the steady state, the capital stock is constant, $K_t = K_{t+1} = K^*$, and determined by

$$K^* = (1-\gamma) \frac{\partial Y(K^*, L)}{\partial L} L. \qquad (2.4.14)$$

All other variables like aggregate consumption, interest rates, wages etc. are constant as well. Consumption when young and when old can differ, as in a setup with finite lifetimes, the interest rate in the steady state does not need to equal the time preference rate of households.

- Transitional dynamics

Dynamics of the capital stock are illustrated in figure 2.4.1. The figure plots the capital stock in period t on the horizontal axis. The capital stock in the next period, K_{t+1}, is plotted on the vertical axis. The law of motion for capital from (2.4.13) then shows up as the curve in this figure. The 45° line equates K_{t+1} to K_t.

We start from our initial condition K_0. Equation (2.4.13) or the curve in this figure then determines the capital stock K_1. This capital stock is then viewed as K_t so that, again, the curve gives us K_{t+1}, which is, given that we now started in 1, the capital stock K_2 of period 2. We can continue doing so and see graphically that the economy approaches the steady state K^* which we computed in (2.4.14).

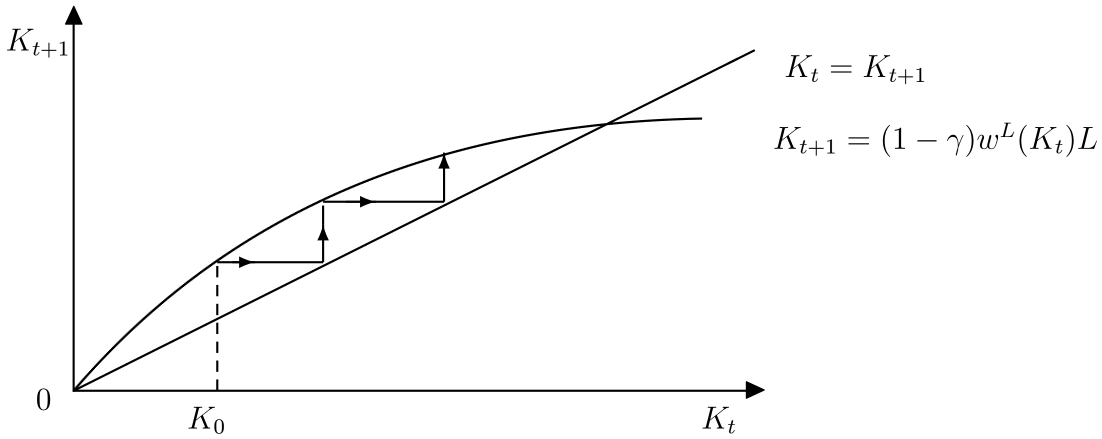


Figure 2.4.1 *Convergence to the steady state*

- Summary

We started with a description of technologies in (2.4.1), preferences in (2.4.6) and factor endowment given by K_0. With behavioural assumptions concerning utility and profit maximization and perfect competition on all markets plus a description of markets in (2.4.3) and some "juggling of equations", we ended up with a one-dimensional difference equation (2.4.13) which describes the evolution of the economy over time and steady state in the long-run. Given this formal analysis of the model, we could now start answering economic questions.

2.5 More on difference equations

The reduced form in (2.4.13) of the general equilibrium model turned out to be a non-linear difference equation. We derived its properties in a fairly intuitive manner. However, one can approach difference equations in a more systematic manner, which we will do in this chapter.

2.5.1 Two useful proofs

Before we look at difference equations, we provide two results on sums which will be useful in what follows. As the proof of this result also has an esthetic value, there will be a second proof of another result to be done in the exercises.

Lemma 2.5.1 *For any $a \neq 1$,*

$$\Sigma_{i=1}^{T} a^i = a\frac{1-a^T}{1-a}, \quad \Sigma_{i=0}^{T} a^i = \frac{1-a^{T+1}}{1-a}$$

Proof. The left hand side is given by

$$\Sigma_{i=1}^{T} a^i = a + a^2 + a^3 + \ldots + a^{T-1} + a^T. \tag{2.5.1}$$

Multiplying this sum by a yields

$$a\Sigma_{i=1}^{T} a^i = a^2 + a^3 + \ldots + a^T + a^{T+1}. \tag{2.5.2}$$

Now subtract (2.5.2) from (2.5.1) and find

$$(1-a)\,\Sigma_{i=1}^{T} a^i = a - a^{T+1} \Leftrightarrow \Sigma_{i=1}^{T} a^i = a\frac{1-a^T}{1-a}. \tag{2.5.3}$$

■

Lemma 2.5.2

$$\Sigma_{i=1}^{T} i a^i = \frac{1}{1-a}\left(a\frac{1-a^T}{1-a} - Ta^{T+1}\right)$$

Proof. The proof is left as exercise 9. ■

2.5.2 A simple difference equation

One of the simplest difference equations is

$$x_{t+1} = a x_t, \quad a > 0. \tag{2.5.4}$$

This equation appears too simple to be worth analysing. We do it here as we get to know the standard steps in analyzing difference equations which we will also use for more complex difference equations. The objective here is therefore not this difference equation as such but what is done with it.

- Solving by substitution

The simplest way to find a solution to (2.5.4) consists of inserting and reinserting this equation sufficiently often. Doing it three times gives

$$x_1 = ax_0,$$
$$x_2 = ax_1 = a^2 x_0,$$
$$x_3 = ax_2 = a^3 x_0.$$

When we look at this solution for $t = 3$ long enough, we see that the general solution is

$$x_t = a^t x_0. \tag{2.5.5}$$

This could formally be proven by either induction or by verification. In this context, we can make use of the following

Definition 2.5.1 *A solution of a difference equation is a function of time which, when inserted into the original difference equation, satisfies this difference equation.*

Equation (2.5.5) gives x as a function of time t only. Verifying that it is a solution indeed just requires inserting it twice into (2.5.4) to see that it satisfies the original difference equation.

- Examples for solutions

The sequence of x_t given by this solution, given different initial conditions x_0, are shown in the following figure for $a > 1$. The parameter values chosen are $a = 2$, $x_0 \in \{0.5, 1, 2\}$ and t runs from 0 to 10.

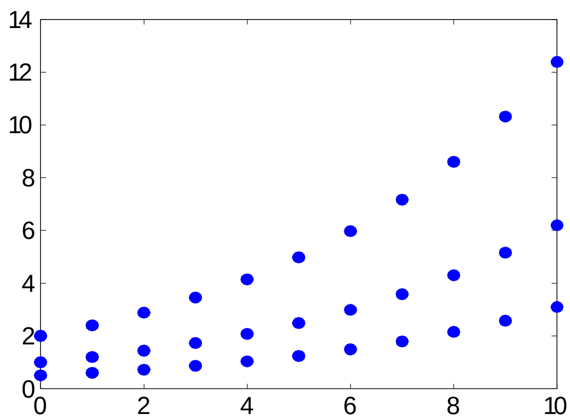


Figure 2.5.1 *Solutions to a difference equation for $a > 1$*

- Long-term behaviour

We can now ask whether x_t approaches a constant when time goes to infinity. This gives

$$\lim_{t\to\infty} x_t = x_0 \lim_{t\to\infty} a^t = \left\{ \begin{array}{c} 0 \\ x_0 \\ \infty \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} 0 < a < 1 \\ a = 1 \\ a > 1 \end{array} \right. .$$

Hence, x_t approaches a constant only when $a < 1$. For $a = 1$, it stays at its initial value x_0.

- A graphical analysis

For more complex difference equations, it often turns out to be useful to analyze their behaviour in a phase diagram. Even though this simple difference equation can be understood easily analytically, we will illustrate its properties in the following figure. Here as well, this allows us to understand how analyses of this type can also be undertaken for more complex difference equations.

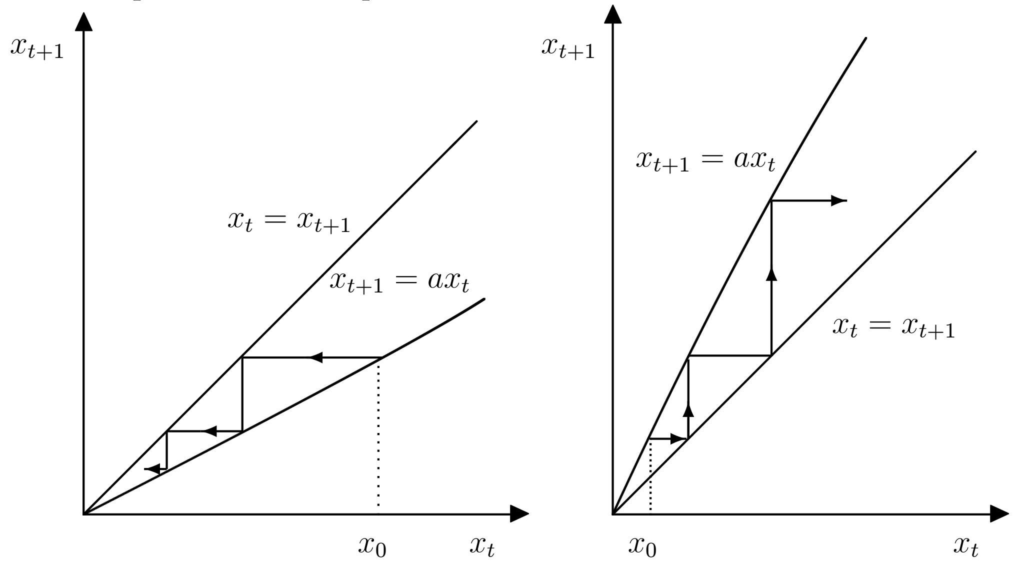


Figure 2.5.2 *A phase diagram for $a < 1$ on the left and $a > 1$ in the right panel*

The principle of a phase diagram is simple. The horizontal axis plots x_t, the vertical axis plots x_{t+1}. There is a 45° line which serves to equate x_t to x_{t+1} and there is a plot of the difference equation we want to understand. In our current example, we plot x_{t+1} as ax_t into the figure. Now start with some initial value x_0 and plot this on the horizontal axis as in the left panel. The value for the next period, i.e. for period 1, can then be read off the vertical axis by looking at the graph of ax_t. This value for x_1 is then copied onto the horizontal axis by using the 45° line. Once on the horizontal axis, we can again use the graph of ax_t to compute the next x_{t+1}. Continuing to do so, the left panel shows how x_t evolves over time, starting at x_0. In this case of $a < 1$, we see how x_t approaches zero. When we graphically illustrate the case of $a > 1$, the evolution of x_t is as shown in the right panel.

2.5.3 A slightly less but still simple difference equation

We now consider a slightly more general difference equation. Compared to (2.5.4), we just add a constant b in each period,

$$x_{t+1} = ax_t + b, \quad a > 0. \tag{2.5.6}$$

We can plot phase diagrams for this difference equation for different parameter values.

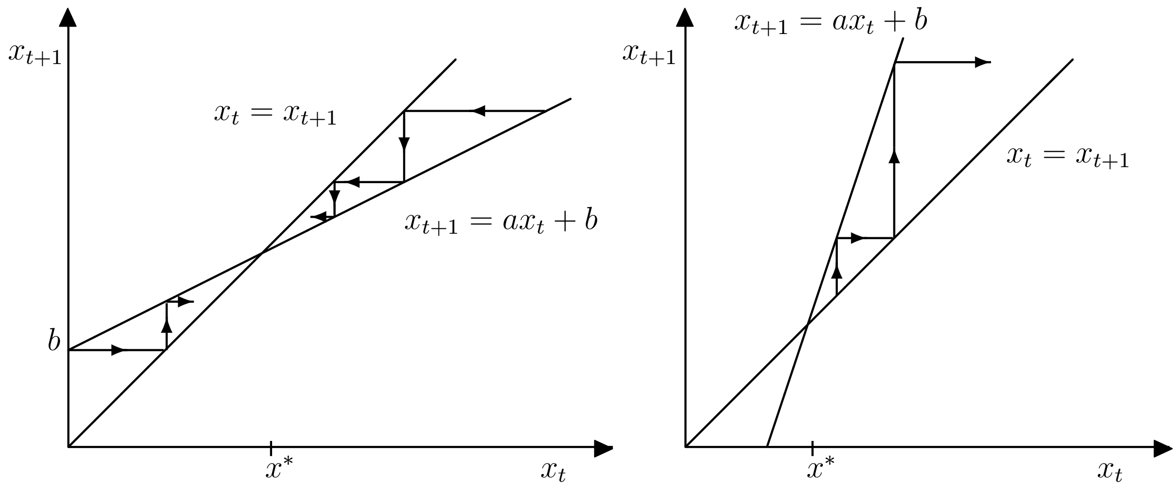


Figure 2.5.3 *Phase diagrams of (2.5.6) for positive (left panel) and negative b (right panel) and higher a in the right panel*

We will work with them shortly.

- Solving by substitution

We solve again by substituting. In contrast to the solution for (2.5.4), we start from an initial value of x_t. Hence, we imagine we are in t (t as today) and compute what the level of x will be tomorrow and the day after tomorrow etc. We find for x_{t+2} and x_{t+3} that

$$x_{t+2} = ax_{t+1} + b = a\left[ax_t + b\right] + b = a^2 x_t + b\left[1 + a\right],$$
$$x_{t+3} = a^3 x_t + b\left[1 + a + a^2\right].$$

This suggests that the general solution is of the form

$$x_{t+n} = a^n x_t + b\Sigma_{i=0}^{n-1} a^i = a^n x_t + b\frac{a^n - 1}{a - 1}.$$

The last equality used the first lemma from ch. 2.5.1.

- Limit for $n \to \infty$ and $a < 1$

The limit for n going to infinity and $a < 1$ is given by

$$\lim_{n\to\infty} x_{t+n} = \lim_{n\to\infty} a^n x_t + b\frac{a^n - 1}{a - 1} = \frac{b}{1-a}. \tag{2.5.7}$$

- A graphical analysis

The left panel in fig. 2.5.3 studies the evolution of x_t for the stable case, i.e. where $0 < a < 1$ and $b > 0$. Starting today in t with x_t, we end up in x^*. As we chose a smaller than one and a positive b, x^* is positive as (2.5.7) shows. We will return to the right panel in a moment.

2.5.4 Fix points and stability

- Definitions

We can use these examples to define two concepts that will also be useful at later stages.

Definition 2.5.2 *(Fixpoint) A fixpoint x^* of a function $f(x)$ is defined by*

$$x^* = f(x^*). \tag{2.5.8}$$

For difference equations of the type $x_{t+1} = f(x_t)$, the fixpoint x^* of the function $f(x_t)$ is also the point where x stays constant, i.e. $x_{t+1} = x_t$. This is usually called the long-run equilibrium point of some economy or its steady or stationary state. Whenever an economic model, represented by its reduced form, is analyzed, it is generally useful to first try and find out whether fixpoints exist and what their economic properties are. For the difference equation from the last section, we obtain

$$x_{t+1} = x_t \equiv x^* \Leftrightarrow x^* = ax^* + b \Leftrightarrow x^* = \frac{b}{1-a}.$$

Once a fixpoint has been identified, one can ask whether it is stable.

Definition 2.5.3 *(Global stability) A fixpoint x^* is globally stable if, starting from an initial value $x_0 \neq x^*$, x_t converges to x^*.*

The concept of global stability usually refers to initial values x_0 that are economically meaningful. An initial physical capital stock that is negative would not be considered to be economically meaningful.

Definition 2.5.4 *(Local stability and instability) A fixpoint x^* is* $\left\{ \begin{array}{c} \textit{unstable} \\ \textit{locally stable} \end{array} \right\}$ *if, starting from an initial value $x^* + \varepsilon$, where ε is small, x_t* $\left\{ \begin{array}{c} \textit{diverges from} \\ \textit{converges to} \end{array} \right\}$ *x^*.*

For illustration purposes consider the fixpoint x^* in the left panel of fig. 2.5.3 - it is globally stable. In the right panel of the same figure, it is unstable. As can easily be seen, the instability follows by simply letting the x_{t+1} line intersect the 45°-line from below. In terms of the underlying difference equation (2.5.6), this requires $b < 0$ and $a > 1$.

Clearly, economic systems can be much more complex and generate several fixpoints. Imagine the link between x_{t+1} and x_t is not linear as in (2.5.6) but nonlinear, $x_{t+1} = f(x_t)$. Unfortunately for economic analysis, a nonlinear relationship is the much more realistic case. The next figure provides an example for some function $f(x_t)$ that implies an unstable x_u^* and a locally stable fixpoint x_s^*.

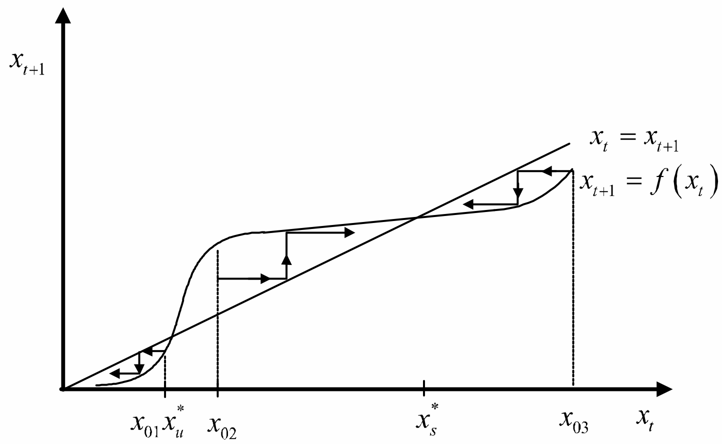


Figure 2.5.4 *A locally stable fixpoint x_s^* and an unstable fixpoint x_u^**

2.5.5 An example: Deriving a budget constraint

A frequently encountered difference equation is the budget constraint. We have worked with budget constraints at various points before but we have hardly thought about their origin. We more or less simply wrote them down. Budget constraints, however, are tricky objects, at least when we think about general equilibrium setups. What is the asset we save in? Is there only one asset or are there several? What are the prices of these assets? How does it relate to the price of the consumption good, i.e. do we express the value of assets in real or nominal terms?

This section will derive a budget constraint. We assume that there is only one asset. The price of one unit of the asset will be denoted by v_t. Its relation to the price p_t of the consumption good will be left unspecified, i.e. we will discuss the most general setup which is possible for the one-asset case. The derivation of a budget constraint is in principle straightforward. One defines the wealth of the household (taking into account which types of assets the household can hold for saving purposes and what their prices are), computes the difference between wealth "today" and "tomorrow" (this is where the difference equation aspect comes in) and uses an equation which relates current savings to current changes in the number of assets. In a final step, one will naturally find out how the interest rate appearing in budget constraints relates to more fundamental quantities like value marginal products and depreciation rates.

- A real budget constraint

The budget constraint which results depends on the measurement of wealth. We start with the case where we measure wealth in units of share, or "number of machines" k_t. Savings of a household who owns k_t shares are given by capital income (net of depreciation losses) plus labour income minus consumption expenditure,

$$s_t \equiv w_t^K k_t - \delta v_t k_t + w_t^L - p_t c_t.$$

This is an identity resulting from bookkeeping of flows at the household level. Savings in t are used for buying new assets in t for which the period-t price v_t needs to be paid,

$$\frac{s_t}{v_t} = k_{t+1} - k_t. \tag{2.5.9}$$

We can rewrite this equation slightly, which will simplify the interpretation of subsequent results, as

$$k_{t+1} = (1-\delta)\, k_t + \frac{w_t^K k_t + w_t^L - p_t c_t}{v_t}.$$

Wealth in the next period expressed in number of stocks (and hence not in nominal terms) is given by wealth which is left over from the current period, $(1-\delta)\, k_t$, plus new acquisitions of stocks which amount to gross capital plus labour income minus consumption expenditure divided by the price of one stock. Collecting the k_t terms and defining an interest rate

$$r_t \equiv \frac{w_t^K}{v_t} - \delta$$

gives a budget constraint for wealth measured by k_t,

$$k_{t+1} = \left(1 + \frac{w_t^K}{v_t} - \delta\right) k_t + \frac{w_t^L}{v_t} - \frac{p_t}{v_t} c_t = (1 + r_t)\, k_t + \frac{w_t^L}{v_t} - \frac{p_t}{v_t} c_t. \tag{2.5.10}$$

This is a difference equation in k_t but not yet a difference equation in nominal wealth a_t.

Rearranging such that expenditure is on the left- and disposable income on the right-hand side yields

$$p_t c_t + v_t k_{t+1} = v_t k_t + \left(w_t^K - v_t \delta\right) k_t + w_t^L.$$

This equation also lends itself to a simple interpretation: On the left-hand side is total expenditure in period t, consisting of consumption expenditure $p_t c_t$ plus expenditure for buying the number of capital goods, k_{t+1}, the household wants to hold in $t+1$. As this expenditure is made in t, total expenditure for capital goods amounts to $v_t k_{t+1}$. The right-hand side is total disposable income which splits into income $v_t k_t$ from selling all capital "inherited" from the previous period, capital income $\left(w_t^K - v_t \delta\right) k_t$ and labour income w_t^L. This is the form budget constraints are often expressed in capital asset pricing models. Note that this is in principle also a difference equation in k_t.

- A nominal budget constraint

In our one-asset case, nominal wealth a_t of a household is given by the number k_t of stocks the household owns (say the number of machines it owns) times the price v_t of one stock (or machine), $a_t = v_t k_t$. Computing the first difference yields

$$\begin{aligned} a_{t+1} - a_t &= v_{t+1} k_{t+1} - v_t k_t \\ &= v_{t+1} \left(k_{t+1} - k_t\right) + \left(v_{t+1} - v_t\right) k_t, \end{aligned} \tag{2.5.11}$$

where the second line added $v_{t+1} k_t - v_{t+1} k_t$. Wealth changes depend on the acquisition $v_{t+1} \left(k_{t+1} - k_t\right)$ of new assets and on changes in the value of assets that are already held, $\left(v_{t+1} - v_t\right) k_t$. Inserting (2.5.9) into (2.5.11) yields

$$\begin{aligned} a_{t+1} - a_t &= \frac{v_{t+1}}{v_t} s_t + \left(v_{t+1} - v_t\right) k_t \\ &= \frac{v_{t+1}}{v_t} \left(\frac{w_t^K}{v_t} a_t - \delta a_t + w_t^L - p_t c_t \right) + \left(\frac{v_{t+1}}{v_t} - 1 \right) a_t \Leftrightarrow \\ a_{t+1} &= \frac{v_{t+1}}{v_t} \left(\left(1 + \frac{w_t^K}{v_t} - \delta \right) a_t + w_t^L - p_t c_t \right). \end{aligned} \tag{2.5.12}$$

What does this equation tell us? Each unit of wealth a_t (say Euro, Dollar, Yen ...) gives $1 - \delta$ units at the end of the period as $\delta\%$ is lost due to depreciation plus "dividend payments" w_t^K / v_t. Wealth is augmented by labour income minus consumption expenditure. This end-of-period wealth is expressed in wealth a_{t+1} at the beginning of the next period by dividing it through v_t (which gives the number k_t of stocks at the end of the period) and multiplying it by the price v_{t+1} of stocks in the next period. We have thereby obtained a difference equation in a_t.

This general budget constraint is fairly complex, however, which implies that in practice it is often expressed differently. One possibility consists of choosing the capital good as the numéraire good and setting $v_t \equiv 1 \; \forall t$. This simplifies (2.5.12) to

$$a_{t+1} = \left(1 + r_t\right) a_t + w_t^L - p_t c_t. \tag{2.5.13}$$

The simplification in this expression consists also in the definition of the interest rate r_t as $r_t \equiv w_t^K - \delta$.

2.6 Further reading and exercises

Rates of substitution are discussed in many books on Microeconomics; see e.g. Mas-Colell, Whinston and Green (1995) or Varian (1992). The definition of the time preference rate is not very explicit in the literature. An alternative formulation implying the same definition as the one we use here is used by Buiter (1981, p. 773). He defines the pure rate of time preference as "the marginal rate of substitution between consumption" in two periods "when equal amounts are consumed in both periods, minus one." A derivation of the time preference rate for a two-period model is in appendix A.1 of Bossmann, Kleiber and Wälde (2007).

The OLG model goes back to Samuelson. For presentations in textbooks, see e.g. Blanchard and Fischer (1989), Azariadis (1993) or de la Croix and Michel (2002). Applications of OLG models are more than numerous. For an example concerning bequests and wealth distributions, see Bossmann, Kleiber and Wälde (2007). See also Galor and Moav (2006) and Galor and Zeira (1993).

The presentation of the Lagrangian is inspired by Intriligator (1971, p. 28 - 30). Treatments of shadow prices are available in many other textbooks (Dixit, 1989, ch. 4; Intriligator, 1971, ch. 3.3). More extensive treatments of difference equations and the implicit function theorem can be found in many introductory "mathematics for economists" books.

There is an interesting discussion on the empirical relevance of exponential discounting. An early analysis of the implications of non-exponential discounting is by Strotz (1955/56). An overview is provided by Frederick et al. (2002). An analysis using stochastic continuous time methods is by Gong et al. (2007).

Exercises chapter 2
Applied Intertemporal Optimization

Optimal consumption in two-period discrete time models

1. Optimal choice of household consumption
   Consider the following maximization problem,
   $$\max_{c_t,c_{t+1}} U_t = v(c_t) + \frac{1}{1+\rho} v(c_{t+1}) \tag{2.6.1}$$
   subject to
   $$w_t + (1+r)^{-1} w_{t+1} = c_t + (1+r)^{-1} c_{t+1}.$$
   Solve it by using the Lagrangian.
   (a) What is the optimal consumption path?
   (b) Under what conditions does consumption rise?
   (c) Show that the first-order conditions can be written as $u'(c_t)/u'(c_{t+1}) = \beta[1+r]$. What does this equation tell you?

2. Solving by substitution
   Consider the maximization problem of section 2.2.1 and solve it by substitution. Solve the constraint for one of the control variables, insert this into the objective function and compute first-order conditions. Show that the same results as in (2.2.4) and (2.2.5) are obtained.

3. Capital market restrictions
   Now consider the following budget constraint. This is a budget constraint that would be appropriate if you want to study the education decisions of households. The parameter b amounts to schooling costs. Inheritance of this individual under consideration is n.
   $$U_t = \gamma \ln c_t + (1-\gamma) \ln c_{t+1}$$
   subject to
   $$-b + n + (1+r)^{-1} w_{t+1} = c_t + (1+r)^{-1} c_{t+1}.$$
   (a) What is the optimal consumption profile under no capital market restrictions?
   (b) Assume loans for financing education are not available, hence savings need to be positive, $s_t \geq 0$. What is the consumption profile in this case?

4. Optimal investment
   Consider a monopolist investing in its technology. Technology is captured by marginal costs c_t. The chief accountant of the firm has provided the manager of the firm with the following information,

$$\Pi = \pi_1 + R\pi_2, \quad \pi_t = p(x_t)\, x_t - c_t x_t - I_t, \quad c_{t+1} = c_t - f(I_1).$$

   Assume you are the manager. What is the optimal investment sequence I_1, I_2?

5. A particular utility function
   Consider the utility function $U = c_t + \beta c_{t+1}$, where $0 < \beta < 1$. Maximize U subject to an arbitrary budget constraint of your choice. Derive consumption in the first and second period. What is strange about this utility function?

6. Intertemporal elasticity of substitution
   Consider the utility function $U = c_t^{1-\sigma} + \beta c_{t+1}^{1-\sigma}$.

   (a) What is the intertemporal elasticity of substitution?

   (b) How can the definition in (2.2.8) of the elasticity of substitution be transformed into the maybe better known definition

$$\epsilon_{ij} = \frac{d \ln (c_i/c_j)}{d \ln \left(u_{c_j}/u_{c_i}\right)} = \frac{p_j/p_i}{c_i/c_j} \frac{d(c_i/c_j)}{d(p_j/p_i)}?$$

   What does ϵ_{ij} stand for in words?

7. An implicit function
   Consider the constraint $x_2 - x_1^\alpha - x_1 = b$.

   (a) Convince yourself that this implicitly defines a function $x_2 = h(x_1)$. Can the function $h(x_1)$ be made explicit?

   (b) Convince yourself that this implicitly defines a function $x_1 = k(x_2)$. Can the function $k(x_2)$ be made explicit?

   (c) Think of a constraint which does *not* define an implicit function.

8. General equilibrium
   Consider the Diamond model for a Cobb-Douglas production function of the form $Y_t = K_t^\alpha L^{1-\alpha}$ and a logarithmic utility function $u = \ln c_t^y + \beta \ln c_{t+1}^o$.

   (a) Derive the difference equation for K_t.

   (b) Draw a phase diagram.

   (c) What are the steady state consumption level and capital stock?

9. Sums

   (a) Proof the statement of the second lemma in ch. 2.5.1,

   $$\Sigma_{i=1}^{T} i a^{i} = \frac{1}{1-a}\left(a\frac{1-a^{T}}{1-a} - T a^{T+1}\right).$$

   The idea is identical to the first proof in ch. 2.5.1.

   (b) Show that

   $$\Sigma_{s=0}^{k-1} c_4^{k-1-s} \nu^{s} = \frac{c_4^{k} - \nu^{k}}{c_4 - \nu}.$$

   Both parameters obey $0 < c_4 < 1$ and $0 < v < 1$. Hint: Rewrite the sum as $c_4^{k-1} \Sigma_{s=0}^{k-1} (\nu/c_4)^{s}$ and observe that the first lemma in ch. 2.5.1 holds for a which are larger or smaller than 1.

10. Difference equations
    Consider the following linear difference equation system

    $$y_{t+1} = a\, y_t + b, \quad a < 0 < b,$$

    (a) What is the fixpoint of this equation?

    (b) Is this point stable?

    (c) Draw a phase diagram.

Chapter 3

Multi-period models

This chapter looks at decision processes where the time horizon is longer than two periods. In most cases, the planning horizon will be infinity. In such a context, Bellman's optimality principle is very useful. Is it, however, not the only way to solve maximization problems with infinite time horizon? For comparison purposes, we therefore start with the Lagrange approach, as in the last section. Bellman's principle will be introduced afterwards when intuition for the problem and relationships will have been increased.

3.1 Intertemporal utility maximization

3.1.1 The setup

The objective function is given by the utility function of an individual,

$$U_t = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u\left(c_\tau\right), \tag{3.1.1}$$

where again as in (2.2.12)

$$\beta \equiv (1+\rho)^{-1}, \quad \rho > 0 \tag{3.1.2}$$

is the discount factor and ρ is the positive time preference rate. We know this utility function already from the definition of the time preference rate, see (2.2.12). The utility function is to be maximized subject to a budget constraint. The difference to the formulation in the last section is that consumption does not have to be determined for two periods only but for infinitely many. Hence, the individual does not choose one or two consumption levels but an entire path of consumption. This path will be denoted by $\{c_\tau\}$. As $\tau \geq t$, $\{c_\tau\}$ is a short form of $\{c_t, c_{t+1}, ...\}$. Note that the utility function is a generalization of the one used above in (2.1.1), but is assumed to be additively separable. The corresponding two period utility function was used in exercise set 1, cf. equation (2.6.1).

The budget constraint can be expressed in the intertemporal version by

$$\Sigma_{\tau=t}^{\infty} (1+r)^{-(\tau-t)} e_\tau = a_t + \Sigma_{\tau=t}^{\infty} (1+r)^{-(\tau-t)} w_\tau, \tag{3.1.3}$$

where $e_\tau = p_\tau c_\tau$. It states that the present value of expenditure equals current wealth a_t plus the present value of labour income w_τ. Labour income w_τ and the interest rate r are exogenously given to the household, its wealth level a_t is given by history. The only quantity that is left to be determined is therefore the path $\{c_\tau\}$. Maximizing (3.1.1) subject to (3.1.3) is a standard Lagrange problem.

3.1.2 Solving by the Lagrangian

The Lagrangian reads

$$\mathcal{L} = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) + \lambda \left[\Sigma_{\tau=t}^{\infty} (1+r)^{-(\tau-t)} e_\tau - a_t - \Sigma_{\tau=t}^{\infty} (1+r)^{-(\tau-t)} w_\tau \right],$$

where λ is the Lagrange multiplier. First-order conditions are

$$\mathcal{L}_{c_\tau} = \beta^{\tau-t} u'(c_\tau) + \lambda [1+r]^{-(\tau-t)} p_\tau = 0, \quad t \leq \tau < \infty, \tag{3.1.4}$$

$$\mathcal{L}_\lambda = 0, \tag{3.1.5}$$

where the latter is, as in the OLG case, the budget constraint. Again, we have as many conditions as variables to be determined: there are infinitely many conditions in (3.1.4), one for each c_τ and one condition for λ in (3.1.5).

Do these first-order conditions tell us something? Take the first-order condition for period τ and for period $\tau+1$. They read

$$\beta^{\tau-t} u'(c_\tau) = -\lambda [1+r]^{-(\tau-t)} p_\tau,$$

$$\beta^{\tau+1-t} u'(c_{\tau+1}) = -\lambda [1+r]^{-(\tau+1-t)} p_{\tau+1}.$$

Dividing them gives

$$\beta^{-1} \frac{u'(c_\tau)}{u'(c_{\tau+1})} = (1+r) \frac{p_\tau}{p_{\tau+1}} \Leftrightarrow \frac{u'(c_\tau)}{\beta u'(c_{\tau+1})} = \frac{p_\tau}{(1+r)^{-1} p_{\tau+1}}. \tag{3.1.6}$$

Rearranging allows us to see an intuitive interpretation: Comparing the instantaneous gain in utility $u'(c_\tau)$ with the future gain, discounted at the time preference rate, $\beta u'(c_{\tau+1})$, must yield the same ratio as the price p_τ that has to be paid today relative to the price that has to be paid in the future, also appropriately discounted to its present value price $(1+r)^{-1} p_{\tau+1}$. This interpretation is identical to the two-period interpretation in (2.2.6) in ch. 2.2.2. If we normalize prices to unity, (3.1.6) is just the expression we obtained in the solution for the two-period maximization problem in (2.6.1).

3.2 The envelope theorem

We saw how the Lagrangian can be used to solve optimization problems with many time periods. In order to understand how dynamic programming works, it is useful to understand a theorem which is frequently used when employing the dynamic programming method: the envelope theorem.

3.2.1 The theorem

In general, the envelope theorem says

Theorem 3.2.1 *Let there be a function $g(x, y)$. Choose x such that $g(.)$ is maximized for a given y. (Assume $g(.)$ is such that a smooth interior solution exists.) Let $f(y)$ be the resulting function of y,*

$$f(y) \equiv \max_x g(x, y).$$

Then the derivative of f with respect to y equals the partial derivative of g with respect to y, if g is evaluated at that $x = x(y)$ that maximizes g,

$$\frac{d\, f(y)}{d\, y} = \left.\frac{\partial\, g(x, y)}{\partial\, y}\right|_{x=x(y)}.$$

Proof. $f(y)$ is constructed by $\frac{\partial}{\partial x} g(x, y) = 0$. This implies a certain $x = x(y)$, provided that second order conditions hold. Hence, $f(y) = \max_x g(x, y) = g(x(y), y)$. Then, $\frac{d\,f(y)}{d\,y} = \frac{\partial\, g(x(y),y)}{\partial\, x} \frac{d\,x(y)}{d\,y} + \frac{\partial\, g(x(y),y)}{\partial\, y}$. The first term of the first term is zero. ■

3.2.2 Illustration

The plane depicts the function $g(x, y)$. The maximum of this function with respect to x is shown as $\max_x g(x, y)$, which is $f(y)$. Given this figure, it is obvious that the derivative of $f(y)$ with respect to y is the same as the partial derivative of $g(.)$ with respect to y at the point where $g(.)$ has its maximum with respect to x: The partial derivative $\frac{\partial g}{\partial y}$ is the derivative when "going in the direction of y". Choosing the highest point of $g(.)$ with respect to x, this directional derivative must be the same as $\frac{df(y)}{dy}$ at the back of the figure.

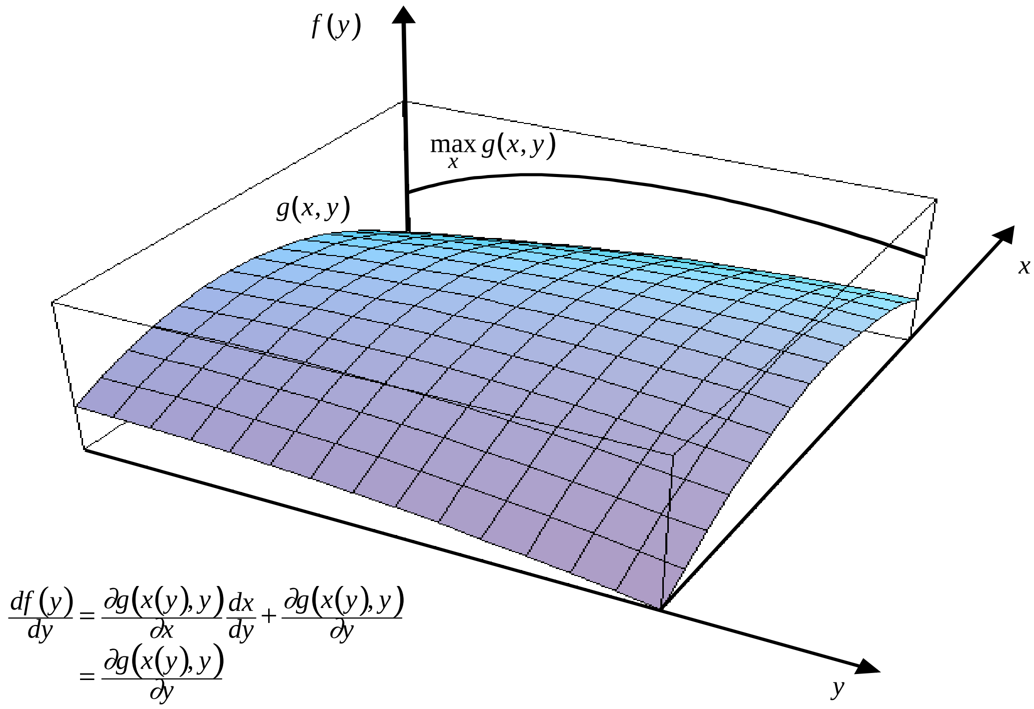


Figure 3.2.1 *Illustrating the envelope theorem*

3.2.3 An example

There is a central planner of an economy. The social welfare function is given by $U(A, B)$, where A and B are consumption goods. The technologies available for producing these goods are $A = A(cL_A)$ and $B = B(L_B)$. The amount of labour used for producing one or the other good is denoted by L_A and L_B and c is a productivity parameter in sector A. The economy's resource constraint is $L_A + L_B = L$.

The planner is interested in maximizing the social welfare level and allocates labour according to $\max_{L_A} U(A(cL_A), B(L - L_A))$. The optimality condition is

$$\frac{\partial U}{\partial A} A' c - \frac{\partial U}{\partial B} B' = 0. \tag{3.2.1}$$

This makes optimal employment L_A a function of c,

$$L_A = L_A(c). \tag{3.2.2}$$

The central planner now asks what happens to the social welfare level when the technology parameter c increases and she still maximizes the social welfare. The latter requires that (3.2.1) continues to hold and the maximized social welfare function with (3.2.2) and the resource constraint reads $U(A(cL_A(c)), B(L - L_A(c)))$. Without using the envelope theorem, the answer is

$$\frac{d}{dc}U(A(cL_A(c)), B(L - L_A(c)))$$
$$= \frac{\partial U(.)}{\partial A}A'[L_A(c) + cL_A'(c)] + \frac{\partial U(.)}{\partial B}B'[-L_A'(c)] = \frac{\partial U(.)}{\partial A}A'L_A(c) > 0,$$

where the last equality follows from inserting the optimality condition (3.2.1). Economically, this result means that the effect of better technology on overall welfare is given by the direct effect in sector A. The indirect effect through the reallocation of labour vanishes as, due to the first-order condition (3.2.1), the marginal contribution of each worker is identical across sectors. Clearly, this only holds for a small change in c.

If one is interested in finding an answer by using the envelope theorem, one would start by defining a function $V(c) \equiv \max_{L_A} U(A(cL_A), B(L - L_A))$. Then, according to the envelope theorem,

$$\begin{aligned} \frac{d}{dc}V(c) &= \frac{\partial}{\partial c}U(A(cL_A), B(L - L_A))\bigg|_{L_A = L_A(c)} \\ &= \frac{\partial U(.)}{\partial A}A'L_A\bigg|_{L_A = L_A(c)} = \frac{\partial U(.)}{\partial A}A'L_A(c) > 0. \end{aligned}$$

Apparently, both approaches yield the same answer. Applying the envelope theorem gives the answer faster.

3.3 Solving by dynamic programming

3.3.1 The setup

We will now get to know how dynamic programming works. Let us study a maximization problem which is similar to the one in ch. 3.1.1. We will use the same utility function as in (3.1.1), reproduced here for convenience, $U_t = \Sigma_{\tau=t}^{\infty}\beta^{\tau-t}u(c_\tau)$. The constraint, however, will be represented in a more general way than in (3.1.3). We stipulate that there is a variable x_t which evolves according to

$$x_{t+1} = f(x_t, c_t). \tag{3.3.1}$$

This variable x_t could represent wealth and this constraint could then represent the budget constraint of the household. This difference equation could also be non-linear, however, as for example in a central planner problem where the constraint is a resource constraint as in (3.9.3). In this case, x_t would stand for capital. Another standard example for x_t as a state variable would be environmental quality. Here we will treat the general case first before we go on to more specific examples further below.

The consumption level c_t and - more generally speaking - other variables whose value is directly chosen by individuals, e.g. investment levels or shares of wealth held in different assets, are called control variables. Variables which are not under the direct control of individuals are called state variables. In many maximization problems, state variables depend indirectly on the behaviour of individuals as in (3.3.1). State variables can also be completely exogenous like for example the TFP level in an exogenous growth model or prices in a household maximization problem.

Optimal behaviour is defined by $\max_{\{c_\tau\}} U_t$ subject to (3.3.1), i.e. the highest value U_t can reach by choosing a sequence $\{c_\tau\} \equiv \{c_t, c_{t+1}, ...\}$ and by satisfying the constraint (3.3.1). The value of this optimal behaviour or optimal program is denoted by

$$V(x_t) \equiv \max_{\{c_\tau\}} U_t \text{ subject to } x_{t+1} = f(x_{t,}c_t). \tag{3.3.2}$$

$V(x_t)$ is called the value function. It is a function of the state variable x_t and not of the control variable c_t. The latter point is easy to understand if one takes into account that the control variable c_t is, when behaving optimally, a function of the state variable x_t.

The value function $V(.)$ could also be a function of time t (e.g. in problems with finite horizon). Generally speaking, x_t and c_t could be vectors and time could then be part of the state vector x_t. The value function is always a function of the states of the system or of the maximization problem.

3.3.2 Three dynamic programming steps

Given this description of the maximization problem, solving by dynamic programming essentially requires us to go through three steps. This three-step approach will be followed here, later in continuous time, and also in models with uncertainty.

- DP1: Bellman equation and first-order conditions

The first step establishes the Bellman equation and computes first-order conditions. The objective function U_t in (3.1.1) is additively separable which means that it can be written in the form

$$U_t = u(c_t) + \beta U_{t+1}. \tag{3.3.3}$$

Bellman's idea consists of rewriting the maximization problem in the optimal program (3.3.2) as

$$V(x_t) \equiv \max_{c_t} \{u(c_t) + \beta V(x_{t+1})\} \tag{3.3.4}$$

subject to

$$x_{t+1} = f(x_t, c_t).$$

Equation (3.3.4) is known as the Bellman equation. In this equation, the problem with potentially infinitely many control variables $\{c_\tau\}$ was broken down in many problems with one control variable c_t. Note that there are *two* steps involved: First, the additive separability of the objective function is used. Second, more importantly, U_{t+1} is replaced by $V(x_{t+1})$. This says that we *assume* that the optimal problem for tomorrow is solved and we should worry about the maximization problem of today only.

We can now compute the first-order condition which is of the form

$$u'(c_t) + \beta V'(x_{t+1}) \frac{\partial f(x_t, c_t)}{\partial c_t} = 0. \tag{3.3.5}$$

It tells us that increasing consumption c_t has advantages and disadvantages. The advantages consist in higher utility today, which is reflected here by marginal utility $u'(c_t)$. The disadvantages come from lower overall utility - the value function V - tomorrow. The reduction in overall utility amounts to the change in x_{t+1}, i.e. the derivative $\partial f(x_t, c_t)/\partial c_t$, times the marginal value of x_{t+1}, i.e. $V'(x_{t+1})$. As the disadvantage arises only tomorrow, this is discounted at the rate β. One can talk of a disadvantage of higher consumption today on overall utility tomorrow as the derivative $\partial f(x_t, c_t)/\partial c_t$ needs to be negative, otherwise an interior solution as assumed in (3.3.5) would not exist.

In principle, this is the solution of our maximization problem. Our control variable c_t is by this expression implicitly given as a function of the state variable, $c_t = c(x_t)$, as x_{t+1} by the constraint (3.3.1) is a function of x_t and c_t. As all state variables in t are known, the control variable is determined by this optimality condition. Hence, as V is well-defined above, we have obtained a solution.

As we know very little about the properties of V at this stage, however, we need to go through two further steps in order to eliminate V (to be precise, its derivative $V'(x_{t+1})$, i.e. the costate variable of x_{t+1}) from this first-order condition and obtain a condition that uses only functions (like e.g. the utility function or the technology for production in later examples) of which properties like signs of first and second derivatives are known. We obtain more information about the evolution of this costate variable in the second dynamic programming step.

- DP2: Evolution of the costate variable

The second step of the dynamic programming approach starts from the "maximized Bellman equation". The maximized Bellman equation is obtained by replacing the control variables in the Bellman equation, i.e. the c_t in (3.3.4), in the present case, by the optimal control variables as given by the first-order condition, i.e. by $c(x_t)$ resulting from (3.3.5). Logically, the max operator disappears (as we insert the $c(x_t)$ which imply a maximum) and the maximized Bellman equation reads

$$V(x_t) = u(c(x_t)) + \beta V(f(x_t, c(x_t))).$$

The derivative with respect to x_t reads

$$V'(x_t) = u'(c(x_t)) \frac{dc(x_t)}{dx_t} + \beta V'(f(x_t, c(x_t))) \left[f_{x_t} + f_c \frac{dc(x_t)}{dx_t} \right].$$

This step shows why it is important that we use the maximized Bellman equation here: Now control variables are a function of state variable and we need to compute the derivative of c_t with respect to x_t when computing the derivative of the value function $V(x_t)$. Inserting the first-order condition simplifies this equation to

$$V'(x_t) = \beta V'(f(x_t, c_t(x_t))) f_{x_t} = \beta V'(x_{t+1}) f_{x_t} \tag{3.3.6}$$

This equation is a difference equation for the costate variable, the derivative of the value function with respect to the state variable, $V'(x_t)$. The costate variable is also called the shadow price of the state variable x_t. If we had more state variables, there would be a costate variable for each state variable. It says how much an additional unit of the state variable (say e.g. of wealth) is valued: As $V(x_t)$ gives the value of optimal behaviour between t and the end of the planning horizon, $V'(x_t)$ says by how much this value changes when x_t is changed marginally. Hence, equation (3.3.6) describes how the shadow price of the state variable changes over time when the agent behaves optimally. If we had used the envelope theorem, we would have immediately ended up with (3.3.6) without having to insert the first-order condition.

- DP3: Inserting first-order conditions

Now insert the first-order condition (3.3.5) twice into (3.3.6) to replace the unknown shadow price and to find an optimality condition depending on u_τ only. This will then be the Euler equation. We do not do this here explicitly as many examples will go through this step in detail in what follows.

3.4 Examples

3.4.1 Intertemporal utility maximization with a CES utility function

The individual's budget constraint is given in the dynamic formulation

$$a_{t+1} = (1 + r_t)(a_t + w_t - c_t). \tag{3.4.1}$$

Note that this dynamic formulation corresponds to the intertemporal version in the sense that (3.1.3) implies (3.4.1) and (3.4.1) with some limit condition implies (3.1.3). This will be shown formally in ch. 3.5.1.

The budget constraint (3.4.1) can be found in many papers and also in some textbooks. The timing as implicit in (3.4.1) is illustrated in the following figure. All events take place at the beginning of the period. Our individual owns a certain amount of wealth a_t at the beginning of t and receives here wage income w_t and spends c_t on consumption also at the beginning. Hence, savings s_t can be used during t for production and interest is paid on s_t which in turn gives a_{t+1} at the beginning of period $t + 1$.

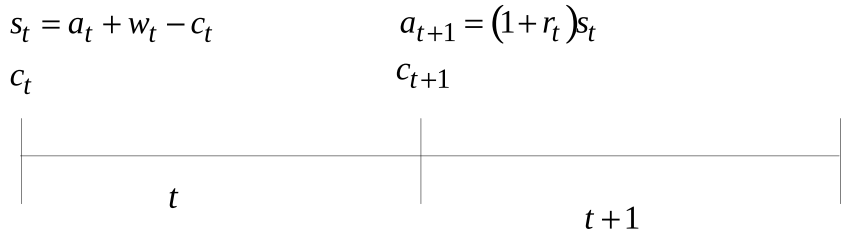


Figure 3.4.1 *The timing in an infinite horizon discrete time model*

The consistency of (3.4.1) with technologies in general equilibrium is not self-evident. We will encounter more conventional budget constraints of the type (2.5.13) further below. As (3.4.1) is widely used, however, we now look at dynamic programming methods and take this budget constraint as given.

The objective of the individual is to maximize her utility function (3.1.1) subject to the budget constraint by choosing a path of consumption levels c_t, denoted by $\{c_\tau\}, \tau \in [t, \infty]$. We will first solve this with a general instantaneous utility function and then insert the CES version of it, i.e.

$$u(c_\tau) = \frac{c_\tau^{1-\sigma} - 1}{1-\sigma}. \tag{3.4.2}$$

The value of the optimal program $\{c_\tau\}$ is, given its initial endowment with a_t, defined as the maximum which can be obtained subject to the constraint, i.e.

$$V(a_t) \equiv \max_{\{c_\tau\}} U_t \tag{3.4.3}$$

subject to (3.4.1). It is called the value function. Its only argument is the state variable a_t. See ch. 3.4.2 for a discussion on state variables and arguments of value functions.

- DP1: Bellman equation and first-order conditions

We know that the utility function can be written as $U_t = u(c_t) + \beta U_{t+1}$. Now assume that the individual behaves optimally as from $t+1$. Then we can insert the value function. The utility function reads $U_t = u(c_t) + \beta V(a_{t+1})$. Inserting this into the value function, we obtain the recursive formulation

$$V(a_t) = \max_{c_t} \{u(c_t) + \beta V(a_{t+1})\}, \tag{3.4.4}$$

known as the Bellman equation.

Again, this breaks down a many-period problem into a two-period problem: The objective of the individual was $\max_{\{c_\tau\}}$ (3.1.1) subject to (3.4.1), as shown by the value function in equation (3.4.3). The Bellman equation (3.4.4), however, is a two period decision problem, the trade-off between consumption today and more wealth tomorrow (under the assumption that the function V is known). This is what is known as Bellman's principle of optimality: Whatever the decision today, subsequent decisions should be made optimally, given the situation tomorrow. History does not count, apart from its impact on the state variable(s).

We now derive a first-order condition for (3.4.4). It reads

$$\frac{d}{dc_t}u\left(c_t\right)+\beta\frac{d}{dc_t}V\left(a_{t+1}\right)=u'\left(c_t\right)+\beta V'\left(a_{t+1}\right)\frac{da_{t+1}}{dc_t}=0.$$

Since $da_{t+1}/dc_t=-\left(1+r_t\right)$ by the budget constraint (3.4.1), this gives

$$u'\left(c_t\right)-\beta\left(1+r_t\right)V'\left(a_{t+1}\right)=0. \tag{3.4.5}$$

Again, this equation makes consumption a function of the state variable, $c_t=c_t\left(a_t\right)$. Following the first-order condition (3.3.5) in the general example, we wrote $c_t=c\left(x_t\right)$, i.e. consumption c_t changes only when the state variable x_t changes. Here, we write $c_t=c_t\left(a_t\right)$, indicating that there can be other variables which can influence consumption other than wealth a_t. An example for such an additional variable in our setup would be the wage rate w_t or interest rate r_t, which after all is visible in the first-order condition (3.4.5). See ch. 3.4.2 for a more detailed discussion of state variables.

Economically, (3.4.5) tells us as before in (3.3.5) that, under optimal behaviour, gains from more consumption today are just balanced by losses from less wealth tomorrow. Wealth tomorrow falls by $1+r_t$, this is evaluated by the shadow price $V'\left(a_{t+1}\right)$ and everything is discounted by β.

- DP2: Evolution of the costate variable

Using the envelope theorem, the derivative of the maximized Bellman equation reads,

$$V'\left(a_t\right)=\beta V'\left(a_{t+1}\right)\frac{\partial a_{t+1}}{\partial a_t}. \tag{3.4.6}$$

We compute the partial derivative of a_{t+1} with respect to a_t as the functional relationship of $c_t=c_t\left(a_t\right)$ should not (because of the envelope theorem) be taken into account. See exercise 2 on p. 71 on a derivation of (3.4.6) without using the envelope theorem.

From the budget constraint we know that $\frac{\partial a_{t+1}}{\partial a_t}=1+r_t$. Hence, the evolution of the shadow price/ the costate variable under optimal behaviour is described by

$$V'\left(a_t\right)=\beta\left[1+r_t\right]V'\left(a_{t+1}\right).$$

This is the analogon to (3.3.6).

- DP3: Inserting first-order conditions

Let us now be explicit about how to insert first-order conditions into this equation. We can insert the first-order condition (3.4.5) on the right-hand side. We can also rewrite the first-order condition (3.4.5), by lagging it by one period, as $\beta(1+r_{t-1})V'(a_t)=u'(c_{t-1})$ and can insert this on the left-hand side. This gives

$$u'(c_{t-1})\beta^{-1}(1+r_{t-1})^{-1}=u'(c_t)\Leftrightarrow u'(c_t)=\beta[1+r_t]u'(c_{t+1}). \tag{3.4.7}$$

This is the same result as the one we obtained when we used the Lagrange method in equation (3.1.6).

It is also the same result as for the two-period saving problem which we found in OLG models - see e.g. (2.2.6) or (2.6.1) in the exercises. This might be surprising as the planning horizons differ considerably between a two- and an infinite-period decision problem. Apparently, whether we plan for two periods or for many more, the change between two periods is always the same when we behave optimally. It should be kept in mind, however, that consumption levels (and not changes) do depend on the length of the planning horizon.

- The CES and logarithmic version of the Euler equation

Let us now insert the CES utility function from (3.4.2) into (3.4.7). Computing marginal utility gives $u'(c_\tau)=c_\tau^{-\sigma}$ and we obtain a linear difference equation in consumption,

$$c_{t+1}=(\beta[1+r_t])^{1/\sigma}c_t. \tag{3.4.8}$$

Note that the logarithmic utility function $u(c_\tau)=\ln c_\tau$, known for the two-period setup from (2.2.1), is a special case of the CES utility function (3.4.2). Letting σ approach unity, we obtain

$$\lim_{\sigma\to1}u(c_\tau)=\lim_{\sigma\to1}\frac{c_\tau^{1-\sigma}-1}{1-\sigma}=\ln c_\tau$$

where the last step used L'Hôspital's rule: The derivative of the numerator with respect to σ is

$$\frac{d}{d\sigma}\left(c_\tau^{1-\sigma}-1\right)=\frac{d}{d\sigma}\left(e^{(1-\sigma)\ln c_\tau}-1\right)=e^{(1-\sigma)\ln c_\tau}(-\ln c_\tau)=c_\tau^{1-\sigma}(-\ln c_\tau).$$

Hence,

$$\lim_{\sigma\to1}\frac{c_\tau^{1-\sigma}-1}{1-\sigma}=\lim_{\sigma\to1}\frac{c_\tau^{1-\sigma}(-\ln c_\tau)}{-1}=\ln c_\tau. \tag{3.4.9}$$

When the logarithmic utility function is inserted into (3.4.7), one obtains an Euler equation as in (3.4.8) with σ set equal to one.

3.4.2 What is a state variable?

Dynamic programming uses the concept of a state variable. In the general version of ch. 3.3, there is clearly only one state variable. It is x_t and its evolution is described in (3.3.1). In the economic example of ch. 3.4.1, the question of what is a state variable is less obvious.

In a strict formal sense, "everything" is a state variable. "Everything" means all variables which are not control variables are state variables. This very broad view of state variables comes from the simple definition that "everything (apart from parameters) which determines control variables is a state variable".

We can understand this view by looking at the explicit solution for the control variables in the two-period example of ch. 2.2.1. We reproduce (2.2.3), (2.2.4) and (2.2.5) for ease of reference,

$$\begin{aligned} W_t &= w_t + (1 + r_{t+1})^{-1} w_{t+1}, \\ c_{t+1} &= (1 - \gamma)(1 + r_{t+1}) W_t, \\ c_t &= \gamma W_t. \end{aligned}$$

We did not use the terms control and state variable there but we could of course solve this two-period problem by dynamic programming as well. Doing so would allow us to understand why "everything" is a state variable. Looking at the solution for c_{t+1} shows that it is a function of r_{t+1}, w_t and w_{t+1}. If we want to make the statement that the control variable is a function of the state variables, then clearly r_{t+1}, w_t and w_{t+1} are state variables.

Generalizing this for our multi-period example from ch. 3.4.1, the entire paths of r_t and w_t are state variables, in addition to wealth a_t. As we are in a deterministic world, we know the evolution of variables r_t and w_t and we can reduce the path of r_t and w_t by the levels of r_t and w_t plus the parameters of the process describing their evolution. Hence, the broad view for state variables applied to ch. 3.4.1 requires us to use r_t, w_t, a_t as state variables.

This broad (and ultimately correct) view of state variables is the reason why the first-order condition (3.4.5) is summarized by $c_t = c_t(a_t)$. The index t captures all variables which influence the solution for c apart from the explicit argument a_t.

In a more practical sense - as opposed to the strict sense - it is highly recommended to consider only the variable which is indirectly affected by the control variable as (the relevant) state variable. Writing the value function as $V = V(a_t, w_t, r_t)$ is possible but highly cumbersome from a notational point of view. What is more, going through the dynamic programming steps does not require more than a_t as a state variable as only the shadow price of a_t is required to obtain an Euler equation and not the shadow price of w_t or r_t. To remind us that more than just a_t has an impact on optimal controls, we should, however, always write $c_t = c_t(a_t)$ as a shortcut for $c_t = c(a_t, w_t, r_t, ...)$.

The conclusion of all this, however, is more cautious: When encountering a new maximization problem and when there is uncertainty about how to solve it and what is a state variable and what is not, it is always the best choice to include more rather than less variables as arguments of the value function. Dropping some arguments afterwards is simpler than adding additional ones.

3.4.3 Optimal R&D effort

In this second example, a research project has to be finished at some future known point in time T. This research project has a certain value at point T and we denote it by D like dissertation. In order to reach this goal, a path of a certain length L needs to be completed. We can think of L as a certain number of pages, a certain number of papers or - probably better - a certain quality of a fixed number of papers. Going through this process of walking and writing is costly, it requires effort e_τ at each point in time $\tau \geq t$. Summing over these cost - think of them as hours worked per day - eventually yields the desired amount of pages,

$$\Sigma_{\tau=t}^{T} f\left(e_\tau\right) = L, \tag{3.4.10}$$

where $f(.)$ is the page of quality production function: More effort means more output, $f'(.) > 0$, but any increase in effort implies a lower increase in output, $f''(.) < 0$.

The objective function of our student is given by

$$U_t = \beta^{T-t} D - \Sigma_{\tau=t}^{T} \beta^{\tau-t} e_\tau. \tag{3.4.11}$$

The value of the completed dissertation is given by D and its present value is obtained by discounting at the discount factor β. Total utility U_t stems from this present value minus the present value of research cost e_τ. The maximization problem consists in maximizing (3.4.11) subject to the constraint (3.4.10) by choosing an effort path $\{e_\tau\}$. The question now arises how these costs are optimally spread over time. How many hours should be worked per day?

An answer can be found by using the Lagrange-approach with (3.4.11) as the objective function and (3.4.10) as the constraint. However, her we will use the dynamic programming approach. Before we can apply it, we need to derive a dynamic budget constraint. We therefore define

$$M_t \equiv \Sigma_{\tau=1}^{t-1} f(e_\tau)$$

as the amount of the pages that have already been written by today. This then implies

$$M_{t+1} - M_t = f(e_t). \tag{3.4.12}$$

The increase in the number of completed pages between today and tomorrow depends on effort-induced output $f\left(e_\tau\right)$ today. We can now apply the three dynamic programming steps.

- DP1: Bellman equation and first-order conditions

The value function can be defined by $V\left(M_t\right) \equiv \max_{\{e_\tau\}} U_t$ subject to the constraint. We follow the approach discussed in ch. 3.4.2 and explicitly use as state variable M_t only, the only state variable relevant for derivations to come. In other words, we explicitly suppress time as an argument of $V(.)$. The reader can go through the derivations by using a value function specified as $V(M_t, t)$ and find out that the same result will obtain. The objective function U_t written recursively reads

$$\begin{aligned} U_t &= \beta\left(\beta^{T-(t+1)}D - \beta^{-1}\Sigma_{\tau=t}^{T}\beta^{\tau-t}e_\tau\right) = \beta\left(\beta^{T-(t+1)}D - \beta^{-1}\left(e_t + \Sigma_{\tau=t+1}^{T}\beta^{\tau-t}e_\tau\right)\right) \\ &= \beta\left(\beta^{T-(t+1)}D - \Sigma_{\tau=t+1}^{T}\beta^{\tau-(t+1)}e_\tau\right) - e_t = \beta U_{t+1} - e_t. \end{aligned}$$

Assuming that the individual behaves optimally as from tomorrow, this reads $U_t = -e_t + \beta V(M_{t+1})$ and the Bellman equation reads

$$V(M_t) = \max_{e_t}\left\{-e_t + \beta V(M_{t+1})\right\}. \tag{3.4.13}$$

The first-order condition is $-1 + \beta V'(M_{t+1})\frac{dM_{t+1}}{de_t} = 0$, which, using the dynamic budget constraint, becomes

$$1 = \beta V'(M_{t+1}) f'(e_t). \tag{3.4.14}$$

Again as in (3.3.5), implicitly and with (3.4.12), this equation defines a functional relationship between the control variable and the state variable, $e_t = e(M_t)$. One unit of additional effort reduces instantaneous utility by 1 but increases the present value of overall utility tomorrow by $\beta V'(M_{t+1}) f'(e_t)$.

- DP2: Evolution of the costate variable

To provide some variation, we will now go through the second step of dynamic programming without using the envelope theorem. Consider the maximized Bellman equation, where we insert $e_t = e(M_t)$ and (3.4.12) into the Bellman equation (3.4.13),

$$V(M_t) = -e(M_t) + \beta V(M_t + f(e(M_t))).$$

The derivative with respect to M_t is

$$\begin{aligned} V'(M_t) &= -e'(M_t) + \beta V'(M_t + f(e(M_t)))\frac{d[M_t + f(e(M_t))]}{dM_t} \\ &= -e'(M_t) + \beta V'(M_{t+1})\left[1 + f'(e(M_t))\,e'(M_t)\right]. \end{aligned}$$

Using the first-order condition (3.4.14) simplifies this derivative to $V'(M_t) = \beta V'(M_{t+1})$. Expressed for $t+1$ gives

$$V'(M_{t+1}) = \beta V'(M_{t+2}) \tag{3.4.15}$$

- DP3: Inserting first-order conditions

The final step inserts the first-order condition (3.4.14) twice to replace $V'(M_{t+1})$ and $V'(M_{t+2})$,

$$\beta^{-1}(f'(e_t))^{-1} = (f'(e_{t+1}))^{-1} \Leftrightarrow \frac{f'(e_{t+1})}{f'(e_t)} = \beta. \tag{3.4.16}$$

The interpretation of this Euler equation is now simple. As $f''(.) < 0$ and $\beta < 1$, effort e_t increases under optimal behaviour, i.e. $e_{t+1} > e_t$. Optimal writing of a dissertation implies more work every day.

- What about levels?

The optimality condition in (3.4.16) specifies only how effort e_τ changes over time, it does not provide information on the level of effort required every day. This is a property of all expressions based on first-order conditions of intertemporal problems. They only give information about changes of levels, not about levels themselves. However, the basic idea for how to obtain information about levels can be easily illustrated.

Assume $f(e_t) = e_t^\gamma$, with $0 < \gamma < 1$. Then (3.4.16) implies (with t being replaced by τ) $e_{\tau+1}^{\gamma-1}/e_\tau^{\gamma-1} = \beta \Leftrightarrow e_{\tau+1} = \beta^{-1/(1-\gamma)} e_\tau$. Solving this difference equation yields

$$e_\tau = \beta^{-(\tau-1)/(1-\gamma)} e_1, \tag{3.4.17}$$

where e_1 is the (at this stage still) unknown initial effort level. Starting in $\tau = 1$ on the first day, inserting this solution into the intertemporal constraint (3.4.10) yields

$$\Sigma_{\tau=1}^T f\left(\beta^{-(\tau-1)/(1-\gamma)} e_1\right) = \Sigma_{\tau=1}^T \beta^{-(\tau-1)\gamma/(1-\gamma)} e_1^\gamma = L.$$

This gives us the initial effort level as (the sum can be solved by using the proofs in ch. 2.5.1)

$$e_1 = \left(\frac{L}{\Sigma_{\tau=1}^T \beta^{-(\tau-1)\gamma/(1-\gamma)}}\right)^{1/\gamma}.$$

With (3.4.17), we have now also computed the level of effort every day.

Behind these simple steps, there is a general principle. Modified first-order conditions resulting from intertemporal problems are difference equations, see for example (2.2.6), (3.1.6), (3.4.7) or (3.4.16) (or differential equations when we work in continuous time later). Any difference (or also differential) equation when solved gives a unique solution only if an initial or terminal condition is provided. Here, we have solved the difference equation in (3.4.16) assuming some initial condition e_1. The meaningful initial condition then followed from the constraint (3.4.10). Hence, in addition to the optimality rule (3.4.16), we always need some additional constraint which allows us to compute the level of optimal behaviour. We return to this point when looking at problems in continuous time in ch. 5.4.

3.5 On budget constraints

We have encountered two different (but related) types of budget constraints so far: dynamic ones and intertemporal ones. Consider the dynamic budget constraint derived in (2.5.13) as an example. Using $e_t \equiv p_t c_t$ for simplicity, it reads

$$a_{t+1} = (1 + r_t) a_t + w_t - e_t. \tag{3.5.1}$$

This budget constraint is called dynamic as it "only" takes what happens between the two periods t and $t+1$,into account. In contrast, an intertemporal budget constraint takes what happens between any starting period (usually t) and the end of the planning horizon into account. In this sense, the intertemporal budget constraint is more comprehensive and contains more information (as we will also see formally below whenever we talk about the no-Ponzi game condition). An example for an intertemporal budget constraint was provided in (3.1.3), replicated here for ease of reference,

$$\Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t)}e_{\tau}=a_t+\Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t)}w_{\tau}. \tag{3.5.2}$$

3.5.1 From intertemporal to dynamic

We will now ask about the link between dynamic and intertemporal budget constraints. Let us choose the simpler link to start with, i.e. the link from the intertemporal to the dynamic version. As an example, take (3.5.2). We will now show that this intertemporal budget constraint implies

$$a_{t+1}=(1+r_t)(a_t+w_t-c_t), \tag{3.5.3}$$

which was used before, for example in (3.4.1).

Write (3.5.2) for the next period as

$$\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t-1)}e_{\tau}=a_{t+1}+\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t-1)}w_{\tau}. \tag{3.5.4}$$

Express (3.5.2) as

$$\begin{aligned}e_t+\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t)}e_{\tau}&=a_t+w_t+\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t)}w_{\tau}\Leftrightarrow\\ e_t+(1+r)^{-1}\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t-1)}e_{\tau}&=a_t+w_t+(1+r)^{-1}\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t-1)}w_{\tau}\Leftrightarrow\\ \Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t-1)}e_{\tau}&=(1+r)(a_t+w_t-e_t)+\Sigma_{\tau=t+1}^{\infty}(1+r)^{-(\tau-t-1)}w_{\tau}.\end{aligned}$$

Insert (3.5.4) and find the dynamic budget constraint (3.5.3).

3.5.2 From dynamic to intertemporal

Let us now ask about the link from the dynamic to the intertemporal budget constraint. How can we obtain the intertemporal version of the budget constraint (3.5.1)?

Technically speaking, this simply requires us to solve a difference equation: In order to solve (3.5.1) recursively, we rewrite it as

$$a_t=\frac{a_{t+1}+e_t-w_t}{1+r_t},\quad a_{t+i}=\frac{a_{t+i+1}+e_{t+i}-w_{t+i}}{1+r_{t+i}}.$$

Inserting sufficiently often yields

$$\begin{aligned}
a_t &= \frac{\frac{a_{t+2}+e_{t+1}-w_{t+1}}{1+r_{t+1}}+e_t-w_t}{1+r_t} = \frac{a_{t+2}+e_{t+1}-w_{t+1}}{(1+r_{t+1})(1+r_t)}+\frac{e_t-w_t}{1+r_t} \\
&= \frac{\frac{a_{t+3}+e_{t+2}-w_{t+2}}{1+r_{t+2}}+e_{t+1}-w_{t+1}}{(1+r_{t+1})(1+r_t)}+\frac{e_t-w_t}{1+r_t} \\
&= \frac{a_{t+3}+e_{t+2}-w_{t+2}}{(1+r_{t+2})(1+r_{t+1})(1+r_t)}+\frac{e_{t+1}-w_{t+1}}{(1+r_{t+1})(1+r_t)}+\frac{e_t-w_t}{1+r_t} \\
&= ... = \lim_{i\to\infty}\frac{a_{t+i}}{(1+r_{t+i-1})\cdots(1+r_{t+1})(1+r_t)}+\Sigma_{i=0}^{\infty}\frac{e_{t+i}-w_{t+i}}{(1+r_{t+i})\cdots(1+r_{t+1})(1+r_t)}.
\end{aligned}$$

The expression in the last line is hopefully instructive but somewhat cumbersome. We can write it in a more concise way as

$$a_t = \lim_{i\to\infty}\frac{a_{t+i}}{\Pi_{s=0}^{i-1}(1+r_{t+s})}+\Sigma_{i=0}^{\infty}\frac{e_{t+i}-w_{t+i}}{\Pi_{s=0}^{i}(1+r_{t+s})}$$

where Π indicates a product, i.e. $\Pi_{s=0}^{i}(1+r_{t+i}) = 1+r_t$ for $i=0$ and $\Pi_{s=0}^{i}(1+r_{t+i}) = (1+r_{t+i})\cdots(1+r_t)$ for $i>0$. For $i=-1$, $\Pi_{s=0}^{i}(1+r_{t+i}) = 1$ by definition.

Letting the limit be zero, a step explained in a second, we obtain

$$a_t = \Sigma_{i=0}^{\infty}\frac{e_{t+i}-w_{t+i}}{\Pi_{s=0}^{i}(1+r_{t+s})} = \Sigma_{\tau=t}^{\infty}\frac{e_\tau-w_\tau}{\Pi_{s=0}^{\tau-t}(1+r_{t+s})} \equiv \Sigma_{\tau=t}^{\infty}\frac{e_\tau-w_\tau}{R_\tau}$$

where the last but one equality is substituted $t+i$ by τ. We can write this as

$$\Sigma_{\tau=t}^{\infty}\frac{e_\tau}{R_\tau} = a_t+\Sigma_{\tau=t}^{\infty}\frac{w_\tau}{R_\tau}. \tag{3.5.5}$$

With a constant interest rate, this reads

$$\Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t+1)}e_\tau = a_t+\Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t+1)}w_\tau. \tag{3.5.6}$$

Equation (3.5.5) is the intertemporal budget constraint that results from a dynamic budget constraint as specified in (3.5.1) using the additional condition that $\lim_{i\to\infty}\frac{a_{t+i}}{\Pi_{s=0}^{i-1}(1+r_{t+s})} = 0$.

Note that the assumption that this limit is zero has a standard economic interpretation. It is usually called the no-Ponzi game condition. To understand the interpretation more easily, just focus on the case of a constant interest rate. The condition then reads $\lim_{i\to\infty} a_{t+i}/(1+r)^i = 0$. The term $a_{t+i}/(1+r)^i$ is the present value in t of wealth a_{t+i} held in $t+i$. The condition says that this present value must be zero.

Imagine an individual that finances expenditure e_τ by increasing debt, i.e. by letting a_{t+i} becoming more and more negative. This condition simply says that an individual's "long-run" debt level, i.e. a_{t+i} for i going to infinity must not increase too quickly - the present value must be zero. Similarly, the condition also requires that an individual should not hold positive wealth in the long run whose present value is not zero. Note that this condition is fulfilled , for example, for any constant debt or wealth level.

3.5.3 Two versions of dynamic budget constraints

Note that we have also encountered two subspecies of dynamic budget constraints. The one from (3.5.1) and the one from (3.5.3). The difference between these two constraints is due to more basic assumptions about the timing of events as was illustrated in fig.s 2.1.1 and 3.4.1.

These two dynamic constraints imply two different versions of intertemporal budget constraints. The version from (3.5.1) leads to (3.5.6) and the one from (3.5.3) leads (with a similar no-Ponzi game condition) to (3.5.2). Comparing (3.5.6) with (3.5.2) shows that the present values on both sides of (3.5.6) discounts one time more than in (3.5.2). The economic difference again lies in the timing, i.e. whether we look at values at the beginning or end of a period.

The budget constraint (3.5.1) is the "natural" budget constraint in the sense that it can be derived easily as above in ch. 2.5.5 and in the sense that it easily aggregates to economy wide resource constraints. We will therefore work with (3.5.1) and the corresponding intertemporal version (3.5.6) in what follows. The reason for not working with them right from the beginning is that the intertemporal version (3.5.2) has some intuitive appeal and that its dynamic version (3.5.3) is widely used in the literature.

3.6 A decentralized general equilibrium analysis

We have so far analyzed maximization problems of households in partial equilibrium. In two-period models, we have analyzed how households can be aggregated and what we learn about the evolution of the economy as a whole. We will now do the same for infinite horizon problems.

As we did in ch. 2.4, we will first specify technologies. This shows what is technologically feasible in this economy. Which goods are produced, which goods can be stored for saving purposes, is there uncertainty in the economy stemming from production processes? Given these technologies, firms maximize profits. Second, household preferences are presented and the budget constraint of households is derived from the technologies presented before. This is the reason why technologies should be presented before households are introduced: budget constraints are endogenous and depend on knowledge of what households can do. Optimality conditions for households are then derived. Finally, aggregation over households and an analysis of properties of the model using the reduced form follows.

3.6.1 Technologies

The technology is a simple Cobb-Douglas technology

$$Y_t = AK_t^{\alpha} L^{1-\alpha}. \tag{3.6.1}$$

Capital K_t and labour L is used with a given total factor productivity level A to produce output Y_t. This good can be used for consumption and investment and equilibrium on the goods market requires

$$Y_t = C_t + I_t. \tag{3.6.2}$$

Gross investment I_t is turned into net investment by taking depreciation into account, $K_{t+1} = (1-\delta) K_t + I_t$. Taking these two equations together gives the resource constraint of the economy,

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t. \tag{3.6.3}$$

As this constraint is simply a consequence of technologies and market clearing, it is identical to the one used in the OLG setup in (2.4.9).

3.6.2 Firms

Firms maximize profits by employing optimal quantities of labour and capital, given the technology in (3.6.1). First-order conditions are

$$\frac{\partial Y_t}{\partial K_t} = w_t^K, \quad \frac{\partial Y_t}{\partial L} = w_t^L \tag{3.6.4}$$

as in (2.4.2), where we have again chosen the consumption good as numéraire.

3.6.3 Households

Preferences of households are described as in the intertemporal utility function (3.1.1). As the only way households can transfer savings from one period to the next is by buying investment goods, an individual's wealth is given by the "number of machines" k_t, she owns. Clearly, adding up all individual wealth stocks gives the total capital stock, $\Sigma_L k_t = K_t$. Wealth k_t increases over time if the household spends less on consumption than what it earns through capital plus labour income, corrected for the loss in wealth each period caused by depreciation,

$$k_{t+1} - k_t = w_t^K k_t - \delta k_t + w_t^L - c_t \Leftrightarrow k_{t+1} = \left(1 + w_t^K - \delta\right) k_t + w_t^L - c_t.$$

If we now define the interest rate to be given by

$$r_t \equiv w_t^K - \delta, \tag{3.6.5}$$

we obtain our budget constraint

$$k_{t+1} = (1 + r_t) k_t + w_t^L - c_t. \tag{3.6.6}$$

Note that the "derivation" of this budget constraint was simplified in comparison to ch. 2.5.5 as the price v_t of an asset is, as we measure it in units of the consumption good which is traded on the same final market (3.6.2), given by 1. More general budget constraints will become pretty complex as soon as the price of the asset is not normalized.

This complexity is needed when it comes e.g. to capital asset pricing - see further below in ch. 9.3. Here, however, this simple constraint is perfect for our purposes.

Given the preferences and the constraint, the Euler equation for this maximization problem is given by (see exercise 5)

$$u'(c_t) = \beta [1 + r_{t+1}] u'(c_{t+1}). \tag{3.6.7}$$

Structurally, this is the same expression as in (3.4.7). The interest rate, however, refers to $t+1$, due to the change in the budget constraint. Remembering that $\beta = 1/(1+\rho)$, this shows that consumption increases as long as $r_{t+1} > \rho$.

3.6.4 Aggregation and reduced form

- Aggregation

To see that individual constraints add up to the aggregate resource constraint, we simply need to take into account that individual income adds up to output, $w_t^K K_t + w_t^L L_t = Y_t$. Remember that we are familiar with the latter from (2.4.4). Now start from (3.6.6) and use (3.6.5) to obtain,

$$K_{t+1} = \Sigma_L k_{t+1} = \left(1 + w_t^K - \delta\right) \Sigma_L k_t + w_t^L L - C_t = (1-\delta) K_t + Y_t - C_t.$$

The optimal behaviour of all households taken together can be gained from (3.6.7) by summing over all households. This is done analytically correctly by first applying the inverse function of u' to this equation and then summing individual consumption levels over all households (see exercise 6 for details). Applying the inverse function again gives

$$u'(C_t) = \beta [1 + r_{t+1}] u'(C_{t+1}), \tag{3.6.8}$$

where C_t is aggregate consumption in t.

- Reduced form

We now need to understand how our economy evolves in general equilibrium. Our first equation is (3.6.8), telling us how consumption evolves over time. This equation contains consumption and the interest rate as endogenous variables.

Our second equation is therefore the definition of the interest rate in (3.6.5) which we combine with the first-order condition of the firm in (3.6.4) to yield

$$r_t = \frac{\partial Y_t}{\partial K_t} - \delta. \tag{3.6.9}$$

This equation contains the interest rate and the capital stock as endogenous variables.

Our final equation is the resource constraint (3.6.3), which provides a link between capital and consumption. Hence, (3.6.8), (3.6.9) and (3.6.3) give a system in three equations and three unknowns. When we insert the interest rate into the optimality condition for consumption, we obtain as our reduced form

$$\begin{aligned} u'(C_t) &= \beta \left[1 + \tfrac{\partial Y_{t+1}}{\partial K_{t+1}} - \delta\right] u'(C_{t+1}), \\ K_{t+1} &= (1-\delta) K_t + Y_t - C_t. \end{aligned} \tag{3.6.10}$$

This is a two-dimensional system of non-linear difference equations which gives a unique solution for the time path of capital and consumption, provided we have two initial conditions K_0 and C_0.

3.6.5 Steady state and transitional dynamics

When trying to understand a system like (3.6.10), the same principles can be followed as with one-dimensional difference equations. First, one tries to identify a fixed point, i.e. a steady state, and then one looks at transitional dynamics.

- Steady state

In a steady state, all variables are constant. Setting $K_{t+1} = K_t = K$ and $C_{t+1} = C_t = C$, we obtain

$$1 = \beta \left(1 + \frac{\partial Y}{\partial K} - \delta\right) \Leftrightarrow \frac{\partial Y}{\partial K} = \rho + \delta, \quad C = Y - \delta K,$$

where the "$\Leftrightarrow$ step" used the link between β and ρ from (3.1.2). In the steady state, the marginal productivity of capital is given by the time preference rate plus the depreciation rate. Consumption equals output minus depreciation, i.e. minus replacement investment. These two equations determine two variables K and C: the first determines K, the second determines C.

- Transitional dynamics

Understanding transitional dynamics is not as straightforward as understanding the steady state. Its analysis follows the same idea as in continuous time, however, and we will analyze transitional dynamics in detail there.

Having said this, we should acknowledge the fact that transitional dynamics in discrete time can quickly become more complex than in continuous time. As an example, chaotic behaviour can occur in one-dimensional difference equations while one needs at least a three-dimensional differential equation system to obtain chaotic properties in continuous time. The literature on chaos theory and textbooks on difference equations provide many examples.

3.7 A central planner

3.7.1 Optimal factor allocation

One of the most solved maximization problems in Economics is the central planner problem. The choice by a central planner given a social welfare function and technological constraints provides information about the first-best factor allocation. This is a benchmark for many analyses in normative economics. We consider the probably most simple case of optimal factor allocation in a dynamic setup.

- The maximization problem

Let preferences be given by

$$U_t = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau), \tag{3.7.1}$$

where C_τ is the aggregate consumption of all households at a point in time τ. This function is maximized subject to a resource accumulation constraint which reads

$$K_{\tau+1} = Y(K_\tau, L_\tau) + (1-\delta) K_\tau - C_\tau \tag{3.7.2}$$

for all $\tau \geq t$. The production technology is given by a neoclassical production function $Y(K_\tau, L_\tau)$ with standard properties.

- The Lagrangian

This setup differs from the ones we got to know before in that there is an infinite number of constraints in (3.7.2). This constraint holds for each point in time $\tau \geq t$. As a consequence, the Lagrangian is formulated with infinitely many Lagrangian multipliers,

$$\mathcal{L} = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau) + \Sigma_{\tau=t}^{\infty} \{\lambda_\tau [K_{\tau+1} - Y(K_\tau, L_\tau) - (1-\delta) K_\tau + C_\tau]\}. \tag{3.7.3}$$

The first part of the Lagrangian is standard, $\Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau)$, it just copies the objective function. The second part consists of a sum from t to infinity, one constraint for each point in time, each multiplied by its own Lagrange multiplier λ_τ. In order to make the maximization procedure clearer, we rewrite the Lagrangian as

$$\begin{aligned}\mathcal{L} = {} & \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau) + \Sigma_{\tau=t}^{s-2} \lambda_\tau [K_{\tau+1} - Y(K_\tau, L_\tau) - (1-\delta) K_\tau + C_\tau] \\ & + \lambda_{s-1} [K_s - Y(K_{s-1}, L_{s-1}) - (1-\delta) K_{s-1} + C_{s-1}] \\ & + \lambda_s [K_{s+1} - Y(K_s, L_s) - (1-\delta) K_s + C_s] \\ & + \Sigma_{\tau=s+1}^{\infty} \lambda_\tau [K_{\tau+1} - Y(K_\tau, L_\tau) - (1-\delta) K_\tau + C_\tau],\end{aligned}$$

where we simply explicitly write out the sum for $s-1$ and s.

Now maximize the Lagrangian both with respect to the control variable C_s, the multipliers λ_s and the state variables K_s. Maximization with respect to K_s might appear unusual at this stage; we will see a justification for this in the next chapter. First-order conditions are

$$\mathcal{L}_{C_s} = \beta^{s-t} u'(C_s) + \lambda_s = 0 \Leftrightarrow \lambda_s = -u'(C_s)\beta^{s-t}, \tag{3.7.4}$$

$$\mathcal{L}_{K_s} = \lambda_{s-1} - \lambda_s \left[\frac{\partial Y}{\partial K_s} + 1 - \delta \right] = 0 \Leftrightarrow \frac{\lambda_{s-1}}{\lambda_s} = 1 + \frac{\partial Y}{\partial K_s} - \delta, \tag{3.7.5}$$

$$\mathcal{L}_{\lambda_s} = 0. \tag{3.7.6}$$

Combining the first and second first-order condition gives $\frac{-u'(C_{s-1})\beta^{s-1-t}}{-u'(C_s)\beta^{s-t}} = 1 + \frac{\partial Y}{\partial K_s} - \delta$. This is equivalent to

$$\frac{u'(C_s)}{\beta u'(C_{s+1})} = \frac{1}{\frac{1}{1+\frac{\partial Y}{\partial K_{s+1}} - \delta}}. \tag{3.7.7}$$

This expression has the same interpretation as (3.1.6) or (3.4.7) for example. When we replace s by τ, this equation with the constraint (3.7.2) is a two-dimensional difference equation system which allows us to determine the paths of capital and consumption, given two boundary conditions, which the economy will follow when factor allocation is optimally chosen. The steady state of such an economy is found by setting $C_s = C_{s+1}$ and $K_s = K_{s+1}$ in (3.7.7) and (3.7.2).

This example also allows us to return to the discussion about the link between the sign of shadow prices and the Lagrange multiplier at the end of ch. 2.3.2. Here, the constraints in the Lagrangian are represented as left-hand side minus right-hand side. As a consequence, the Lagrange multipliers are negative, as the first-order conditions (3.7.4) show. Apart from the fact that the Lagrange multiplier here now stands for *minus* the shadow price, this does not play any role for the final description of optimality in (3.7.7).

3.7.2 Where the Lagrangian comes from II

Let us now see how we can derive the same expression as that in (3.7.7) without using the Lagrangian. This will allow us to give an intuitive explanation for why we maximized the Lagrangian in the last chapter with respect to both the control and the state variable.

- Maximization without Lagrange

Insert the constraint (3.7.2) into the objective function (3.7.1) and find

$$U_t = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u\left(Y(K_\tau, L_\tau) + (1-\delta) K_\tau - K_{\tau+1}\right) \longrightarrow \max_{\{K_\tau\}}$$

This is now maximized by choosing a path $\{K_\tau\}$ for capital. Choosing the state variable implicitly pins down the path $\{C_\tau\}$ of the control variable consumption and one can therefore think of this maximization problem as one where consumption is optimally chosen.

Now rewrite the objective function as

$$\begin{aligned} U_t = & \Sigma_{\tau=t}^{s-2} \beta^{\tau-t} u \left(Y \left(K_\tau, L_\tau \right) + (1-\delta) K_\tau - K_{\tau+1} \right) \\ & + \beta^{s-1-t} u \left(Y \left(K_{s-1}, L_{s-1} \right) + (1-\delta) K_{s-1} - K_s \right) \\ & + \beta^{s-t} u \left(Y \left(K_s, L_s \right) + (1-\delta) K_s - K_{s+1} \right) \\ & + \Sigma_{\tau=s+1}^{\infty} \beta^{\tau-t} u \left(Y \left(K_\tau, L_\tau \right) + (1-\delta) K_\tau - K_{\tau+1} \right). \end{aligned}$$

Choosing K_s optimally implies

$$\begin{aligned} & - \beta^{s-1-t} u' \left(Y \left(K_{s-1}, L_{s-1} \right) + (1-\delta) K_{s-1} - K_s \right) \\ & + \beta^{s-t} u' \left(Y \left(K_s, L_s \right) + (1-\delta) K_s - K_{s+1} \right) \left(\frac{\partial Y \left(K_s, L_s \right)}{\partial K_s} + 1 - \delta \right) = 0. \end{aligned}$$

Reinserting the constraint (3.7.2) and rearranging gives

$$u' \left(C_{s-1} \right) = \beta u' \left(C_s \right) \left(1 + \frac{\partial Y \left(K_s, L_s \right)}{\partial K_s} - \delta \right).$$

This is the standard optimality condition for consumption which we obtained in (3.7.7). As s can stand for any point in time between t and infinity, we could replace s by t, τ or $\tau + 1$.

- Back to the Lagrangian

When we now go back to the maximization procedure where the Lagrangian was used, we see that the partial derivative of the Lagrangian with respect to K_t in (3.7.5) captures how λ_t changes over time. The simple reason why the Lagrangian is maximized with respect to K_t is therefore that an additional first-order condition is needed as λ_t needs to be determined as well.

In static maximization problems with two consumption goods and one constraint, the Lagrangian is maximized by choosing consumption levels for both consumption goods and by choosing the Lagrange multiplier. In the Lagrange setup above in (3.7.4) to (3.7.6), we choose both endogenous variables K_t and C_t plus the multiplier λ_t and thereby determine optimal paths for all three variables. Hence, it is a technical - mathematical - reason that K_t is "chosen": determining three unknowns simply requires three first-order conditions. Economically, however, the control variable C_t is "economically chosen" while the state variable K_t adjusts indirectly as a consequence of the choice of C_t.

3.8 Growth of family size

3.8.1 The setup

Let us now consider an extension to the models considered so far. Let us imagine there is a family consisting of n_τ members at point in time τ and let consumption c_τ per individual family member be optimally chosen by the head of the family. The objective function for this family head consists of instantaneous utility $u(.)$ per family member times the number of members, discounted at the usual discount factor β,

$$U_t = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) n_\tau.$$

Let us denote family wealth by $\hat{a}_t$. It is the product of individual wealth times the number of family members, $\hat{a}_t \equiv n_t a_t$. The budget constraint of the household is then given by

$$\hat{a}_{t+1} = (1+r_t)\hat{a}_t + n_t w_t - n_t c_t$$

Total labour income is given by n_t times the wage w_t and family consumption is $n_t c_t$.

3.8.2 Solving by substitution

We solve this maximization problem by substitution. We rewrite the objective function and insert the constraint twice,

$$\left\{\Sigma_{\tau=t}^{s-1} \beta^{\tau-t} u(c_\tau) n_\tau + \beta^{s-t} u(c_s) n_s + \beta^{s+1-t} u(c_{s+1}) n_{s+1} + \Sigma_{\tau=s+2}^{\infty} \beta^{\tau-t} u(c_\tau) n_\tau\right\}$$
$$= \Sigma_{\tau=t}^{s-1} \beta^{\tau-t} u(c_\tau) n_\tau + \beta^{s-t} u\left(\frac{(1+r_s)\hat{a}_s + n_s w_s - \hat{a}_{s+1}}{n_s}\right) n_s$$
$$+ \beta^{s+1-t} u\left(\frac{(1+r_{s+1})\hat{a}_{s+1} + n_{s+1} w_{s+1} - \hat{a}_{s+2}}{n_{s+1}}\right) n_{s+1} + \Sigma_{\tau=s+2}^{\infty} \beta^{\tau-t} u(c_\tau) n_\tau.$$

Now compute the derivative with respect to $\hat{a}_{s+1}$. This gives

$$u'\left(\frac{(1+r_s)\hat{a}_s + n_s w_s - \hat{a}_{s+1}}{n_s}\right)\frac{n_s}{n_s} = \beta u'\left(\frac{(1+r_{s+1})\hat{a}_{s+1} + n_{s+1} w_{s+1} - \hat{a}_{s+2}}{n_{s+1}}\right)\frac{1+r_{s+1}}{n_{s+1}} n_{s+1}.$$

When we replace the budget constraint by consumption again and cancel the n_s and n_{s+1}, we obtain

$$u'(c_s) = \beta\left[1+r_{s+1}\right] u'(c_{s+1}). \tag{3.8.1}$$

The interesting feature of this rule is that being part of a family whose size n_τ can change over time does not affect the growth of individual consumption c_s. It follows the same rule as if individuals maximized utility independently of each other and with their personal budget constraints.

3.8.3 Solving by the Lagrangian

The Lagrangian for this setup with one budget constraint for each point in time requires an infinite number of Lagrange multipliers λ_τ, one for each τ. It reads

$$\mathcal{L} = \Sigma_{\tau=t}^{\infty}\left\{\beta^{\tau-t} u(c_\tau) n_\tau + \lambda_\tau\left[(1+r_\tau)\hat{a}_\tau + n_\tau l_\tau w_\tau - n_\tau c_\tau - \hat{a}_{\tau+1}\right]\right\}.$$

We first compute the first-order conditions for consumption and hours worked for one point in time s,

$$\mathcal{L}_{c_s} = \beta^{s-t} \frac{\partial}{\partial c_s} u(c_s) - \lambda_s = 0,$$

As discussed in 3.7, we also need to compute the derivative with respect to the state variable. It is important to compute the derivative with respect to family wealth $\hat{a}_\tau$ as this is the true state variable of the head of the family. (Computing the derivative with respect to individual wealth a_τ would also work but would lead to an incorrect result, i.e. a result that differs from 3.8.1.) This derivative is

$$\mathcal{L}_{\hat{a}_s} = -\lambda_{s-1} + \lambda_s (1 + r_s) = 0 \Leftrightarrow \lambda_{s-1} = (1 + r_s) \lambda_s.$$

Optimal consumption then follows by replacing the Lagrange multipliers, $u'(c_{s-1}) = \beta [1 + r_s] u'(c_s)$. This is identical to the result we obtain by inserting in (3.8.1).

3.9 Further reading and exercises

For a much more detailed background on the elasticity of substitution, see Blackorby and Russell (1989). They study the case of more than two inputs and stress that the Morishima elasticity is to be preferred to the Allan/ Uzawa elasticity.

The dynamic programming approach was developed by Bellman (1957). Maximization using the Lagrange method is widely applied by Chow (1997). The example in ch. 3.4.3 was originally inspired by Grossman and Shapiro (1986).

Exercises chapter 3
Applied Intertemporal Optimization

Dynamic programming in discrete deterministic time

1. The envelope theorem I
   Let the utility function of an individual be given by
   $$U = U(C, L),$$
   where consumption C increases utility and supply of labour L decreases utility. Let the budget constraint of the individual be given by
   $$wL = C.$$
   Let the individual maximize utility with respect to consumption and the amount of labour supplied.
   (a) What is the optimal labour supply function (in implicit form)? How much does an individual consume? What is the indirect utility function?
   (b) Under what conditions does an individual increase labour supply when wages rise (no analytical solution required)?
   (c) Assume higher wages lead to increased labour supply. Does disutility arising from increased labour supply compensate utility from higher consumption? Does utility rise if there is no disutility from working? Start from the indirect utility function derived in a) and apply the proof of the envelope theorem and the envelope theorem itself.

2. The envelope theorem II
   (a) Compute the derivative of the Bellman equation (3.4.6) without using the envelope theorem. Hint: Compute the derivative with respect to the state variable and then insert the first-order condition.
   (b) Do the same with (3.4.15)

3. The additively separable objective function
   (a) Show that the objective function can be written as in (3.3.3).

(b) Find out whether (3.3.3) implies the objective function. (It does not.)

4. Intertemporal and dynamic budget constraints

(a) Show that the intertemporal budget constraint

$$\Sigma_{\tau=t}^{T}\left(\Pi_{k=t}^{\tau-1}\frac{1}{1+r_k}\right)e_\tau = a_t + \Sigma_{\tau=t}^{T}\left(\Pi_{k=t}^{\tau-1}\frac{1}{1+r_k}\right)i_\tau \tag{3.9.1}$$

implies the dynamic budget constraint

$$a_{t+1} = (a_t + i_t - e_t)(1+r_t)\,. \tag{3.9.2}$$

(b) Under which conditions does the dynamic budget constraint imply the intertemporal budget constraint?

(c) Now consider $a_{t+1} = (1+r_t)\,a_t + w_t - e_t$. What intertemporal budget constraint does it imply?

5. The standard saving problem
Consider the objective function from (3.1.1), $U_t = \Sigma_{\tau=t}^{\infty}\beta^{\tau-t}u(c_\tau)$, and maximize it by choosing a consumption path $\{c_\tau\}$ subject to the constraint (3.6.6), $k_{t+1} = (1+r_t)\,k_t + w_t^L - c_t$. The result is given by (3.6.7).

(a) Solve this problem by dynamic programming methods.

(b) Solve this by using the Lagrange approach. Choose a multiplier λ_t for an infinite sequence of constraints.

(c) Solve this by using the Lagrange approach and construct an intertemporal budget constraint beforehand.

6. Aggregation of optimal consumption rules
Consider the optimality condition $u'(c_t) = \beta(1+r_{t+1})\,u'(c_{t+1})$ in (3.6.7) and derive the aggregate version (3.6.8). Find the assumptions required for the utility function for these steps to be possible.

7. A benevolent central planner
You are the omniscient omnipotent benevolent central planner of an economy. You want to maximize a social welfare function

$$U_t = \Sigma_{\tau=t}^{\infty}\beta^{\tau-t}u(C_\tau)$$

for your economy by choosing a path of aggregate consumption levels $\{C_\tau\}$ subject to a resource constraint

$$K_{t+1} - K_t = Y(K_t, L_t) - \delta\, K_t - C_t \tag{3.9.3}$$

(a) Solve this problem by dynamic programming methods.

(b) Discuss how the central planner result is related to the decentralized result from exercise 5.

(c) What does the result look like for a utility function which is logarithmic and for one which has constant elasticity of substitution,

$$u(C(\tau)) = \phi \ln C(\tau) \text{ and } u(C(\tau)) = \frac{C(\tau)^{1-\sigma} - 1}{1-\sigma}? \tag{3.9.4}$$

8. Environmental economics
Imagine you are an economist only interested in maximizing the present value of your endowment. You own a renewable resource, for example a piece of forest. The amount of wood in your forest at a point in time t is given by x_t. Trees grow at $b(x_t)$ and you harvest at t the quantity c_t.

   (a) What is the law of motion for x_t?

   (b) What is your objective function if prices at t per unit of wood is given by p_t, your horizon is infinity and you have perfect information?

   (c) How much should you harvest per period when the interest rate is constant? Does this change when the interest rate is time-variable?

9. The 10k run - Formulating and solving a maximization problem
You consider participation in a 10k run or a marathon. The event will take place in M months. You know that your fitness needs to be improved and that this will be costly: it requires effort a_0 which reduces utility $u(.)$. At the same time, you enjoy being fast, i.e. utility increases the shorter your finish time l. The higher your effort, the shorter your finish time.

   (a) Formulate a maximization problem with 2 periods. Effort affects the finish time in M months. Specify a utility function and discuss a reasonable functional form which captures the link between finish time l and effort a_0.

   (b) Solve this maximization problem by providing and discussing the first-order condition.

10. A central planner
Consider the objective function of a central planner,

$$U_0 = \Sigma_{t=0}^{\infty} \beta^t u(C_t). \tag{3.9.5}$$

The constraint is given by

$$K_{t+1} = Y(K_t, L_t) + (1-\delta) K_t - C_t. \tag{3.9.6}$$

(a) Explain in words the meaning of the objective function and of the constraint.

(b) Solve the maximization problem by first inserting (3.9.6) into (3.9.5) and then by optimally choosing K_t. Show that the result is $\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + \frac{\partial Y(K_{t+1}, L_{t+1})}{\partial K_{t+1}} - \delta$ and discuss this in words.

(c) Discuss why using the Lagrangian also requires maximizing with respect to K_t even though K_t is a state variable.

Part II

Deterministic models in continuous time

Part II covers continuous time models under certainty. Chapter 4 first looks at differential equations as they are the basis of the description and solution of maximization problems in continuous time. First, some useful definitions and theorems are provided. Second, differential equations and differential equation systems are analyzed qualitatively by the so-called "phase-diagram analysis". This simple method is extremely useful for understanding differential equations per se and also for later purposes for understanding qualitative properties of solutions to maximization problems and properties of whole economies. Linear differential equations and their economic applications are then finally analyzed before some words are spent on linear differential equation systems.

Chapter 5 presents a new method for solving maximization problems - the Hamiltonian. As we are now in continuous time, two-period models do not exist. A distinction will be drawn, however, between finite and infinite horizon models. In practice, this distinction is not very important as, as we will see, optimality conditions are very similar for finite and infinite maximization problems. After an introductory example on maximization in continuous time by using the Hamiltonian, the simple link between Hamiltonians and the Lagrangian is shown.

The solution to maximization problems in continuous time will consist of one or several differential equations. As a unique solution to differential equations requires boundary conditions, we will show how boundary conditions are related to the type of maximization problem analyzed. The boundary conditions differ significantly between finite and infinite horizon models. For the finite horizon models, there are initial or terminal conditions. For the infinite horizon models, we will get to know the transversality condition and other related conditions like the No-Ponzi-game condition. Many examples and a comparison between the present-value and the current-value Hamiltonian conclude this chapter.

Chapter 6 solves the same kind of problems as chapter 5, but it uses the method of "dynamic programming". The reason for doing this is to simplify understanding of dynamic programming in stochastic setups in Part IV. Various aspects specific to the use of dynamic programming in continuous time, e.g. the structure of the Bellman equation, can already be treated here under certainty. This chapter will also provide a comparison between the Hamiltonian and dynamic programming and look at a maximization problem with two state variables. An example from monetary economics on real and nominal interest rates concludes the chapter.

Chapter 4

Differential equations

There are many excellent textbooks on differential equations. This chapter will therefore be relatively short. Its objective is more to recap basic concepts taught in other courses and to serve as a background for later applications.

4.1 Some definitions and theorems

4.1.1 Definitions

The following definitions are standard and follow Brock and Malliaris (1989).

Definition 4.1.1 *An ordinary differential equation system (ODE system) is of the type*

$$\frac{dx(t)}{dt} \equiv \dot{x}(t) = f(t, x(t)), \tag{4.1.1}$$

where t lies between some starting point and infinity, $t \in [t_0, \infty[$, x can be a vector, $x \in R^n$ and f maps from R^{n+1} into R^n. When x is not a vector, i.e. for $n = 1$, (4.1.1) obviously is an ordinary differential equation.

An autonomous differential equation is an ordinary differential equation where $f(.)$ is independent of time t,

$$\frac{dx(t)}{dt} \equiv \dot{x}(t) = f(x(t)). \tag{4.1.2}$$

The difference between a differential equation and a "normal" algebraic equation obviously lies in the fact that differential equations contain derivatives of variables like $\dot{x}(t)$. An example of a differential equation which is not an ODE is the partial differential equation. A linear example is

$$a(x,t)\frac{\partial p(x,t)}{\partial x} + b(x,t)\frac{\partial p(x,t)}{\partial t} = c(x,t),$$

where $a(.)$, $b(.)$, $c(.)$ and $p(.)$ are functions with "nice properties". While in an ODE, there is one derivative (often with respect to time), a partial differential equation contains derivatives with respect to several variables. Partial differential equations can describe e.g. a density and how it changes over time. Other types of differential equations include stochastic differential equations (see ch. 9), implicit differential equations (which are of the type $g(\dot{x}(t)) = f(t, x(t))$), delay differential equations ($\dot{x}(t) = f(x(t-\Delta))$) and many other more.

Definition 4.1.2 *An initial value problem is described by*

$$\dot{x} = f(t,x), \quad x(t_0) = x_{0,} \quad t \in [t_0, T],$$

where x_0 is the initial condition.

A terminal value problem is of the form

$$\dot{x} = f(t,x), \quad x(T) = x_T, \quad t \in [t_0, T],$$

where x_T is the terminal condition.

4.1.2 Two theorems

Theorem 4.1.1 *Existence (Brock and Malliaris, 1989)*
If $f(t,x)$ is a continuous function on rectangle $L = \{(t,x) | \quad |t - t_0| \leq a, |x - x_0| \leq b\}$ then there exists a continuous differentiable solution $x(t)$ on interval $|t - t_0| \leq a$ that solves initial value problem

$$\dot{x} = f(t,x), \quad x(t_0) = x_0. \tag{4.1.3}$$

This theorem only proves that a solution exists. It is still possible that there are many solutions.

Theorem 4.1.2 *Uniqueness*
If f and $\partial f / \partial x$ are continuous functions on L, the initial value problem (4.1.3) has a unique solution for

$$t \in \left[t_0, t_0 + \min\left\{a, \frac{b}{\max|f(t,x)|}\right\}\right]$$

If this condition is met, an ODE with an initial or terminal condition has a unique solution. More generally speaking, a differential equation system consisting of n ODEs that satisfy these conditions (which are met in the economic problems we encounter here) has a unique solution provided that there are n boundary conditions. Knowing about a unique solution is useful as one knows that changes in parameters imply unambiguous predictions about changes in endogenous variables. If the government changes some tax, we can unambiguously predict whether employment goes up or down.

4.2 Analyzing ODEs through phase diagrams

This section will present tools that allow us to determine properties of solutions of differential equations and differential equation systems. The analysis will be qualitative in this chapter as most economic systems are too non-linear to allow for an explicit general analytic solution. Explicit solutions for linear differential equations will be treated in ch. 4.3.

4.2.1 One-dimensional systems

We start with a one-dimensional differential equation $\dot{x}(t) = f(x(t))$, where $x \in R$ and $t > 0$. This will also allow us to review the concepts of fixpoints, local and global stability and instability as used already when analyzing difference equations in ch. 2.5.4.

- Unique fixpoint

Let $f(x)$ be represented by the graph in the following figure, with $x(t)$ being shown on the horizontal axis. As $f(x)$ gives the change of $x(t)$ over time, $\dot{x}(t)$ is plotted on the vertical axis.

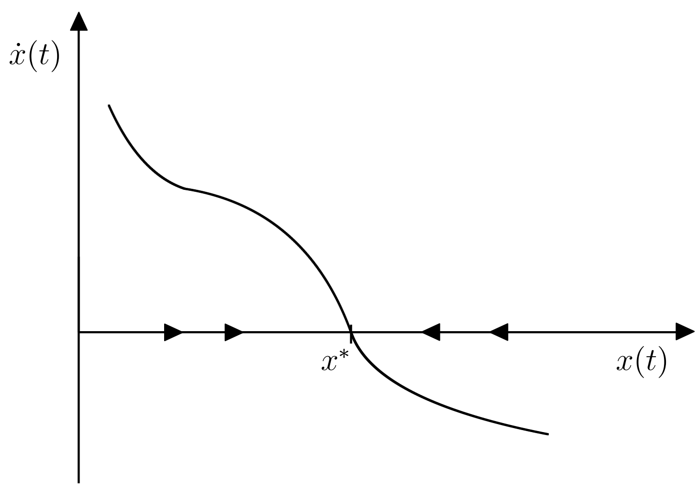


Figure 4.2.1 *Qualitative analysis of a differential equation*

As in the analysis of difference equations in ch. 2.5.4, we first look for the fixpoint of the underlying differential equation. A fixpoint is defined similarly in spirit but - thinking now in continuous time - differently in detail.

Definition 4.2.1 *A fixpoint x^* is a point where $x(t)$ does not change. In continuous time, this means $\dot{x}(t) = 0$ which, from the definition (4.1.2) of the differential equation, requires $f(x^*) = 0$.*

The requirement that $x(t)$ does not change is the similarity in spirit to the definition in discrete time. The requirement that $f(x^*) = 0$ is the difference in detail: in discrete time as in (2.5.8) we required $f(x^*) = x^*$. Looking at the above graph of $f(x)$, we find x^* at the point where $f(x)$ crosses the horizontal line.

We then inquire the stability of the fixpoint. When x is to the left of x^*, $f(x) > 0$ and therefore x increases, $\dot{x}(t) > 0$. This increase of x is represented in this figure by the arrows on the horizontal axis. Similarly, for $x > x^*$, $f(x) < 0$ and $x(t)$ decreases. We have therefore found that the fixpoint x^* is globally stable and have also obtained a "feeling" for the behaviour of $x(t)$, given some initial conditions.

We can now qualitatively plot the solutions with time t on the horizontal axis. As the discussion has just shown, the solution $x(t)$ depends on the initial value from which we start, i.e. on $x(0)$. For $x(0) > x^*$, $x(t)$ decreases, for $x(0) < x^*$, $x(t)$ increases: any changes over time are monotonic. There is one solution for each initial condition. The following figure shows three solutions of $\dot{x}(t) = f(x(t))$, given three different initial conditions.

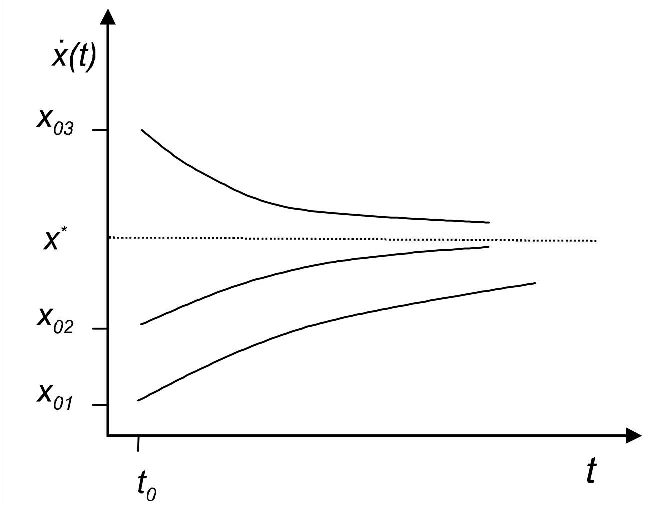


Figure 4.2.2 *Qualitative solutions of an ODE for three different initial conditions*

- Multiple fixpoints and equilibria

Of course, more sophisticated functions than $f(x)$ can be imagined. Now consider a differential equation $\dot{x}(t) = g(x(t))$ where $g(x)$ is non-monotonic as plotted in the next figure.

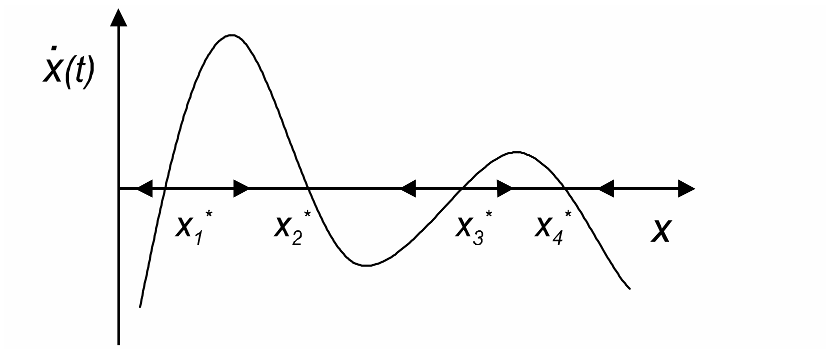


Figure 4.2.3 *Multiple equilibria*

As this figure shows, there are four fixpoints. Looking at whether $g(x)$ is positive or negative, we know whether $x(t)$ increases or decreases over time. This allows us to plot arrows on the horizontal axis as in the previous example. The difference to before consists of the fact that some fixpoints are unstable and some are stable. Definition 4.2.1 showed us that the concept of a fixpoint in continuous time is slightly different from discrete time. However, the definitions of stability as they were introduced in discrete time can be directly applied here as well. Looking at x_1^*, any small deviation of x from x_1^* implies an increase or decrease of x. The fixpoint x_1^* is therefore unstable, given the definition in ch. 2.5.4. Any small deviation x_2^*, however, implies that x moves back to x_2^*. Hence, x_2^* is (locally) stable. The fixpoint x_3^* is also unstable, while x_4^* is again locally stable: While x converges to x_4^* for any $x > x_4^*$ (in this sense x_4^* could be called globally stable from the right), x converges to x_4^* from the left only if x is not smaller than or equal to x_3^*.

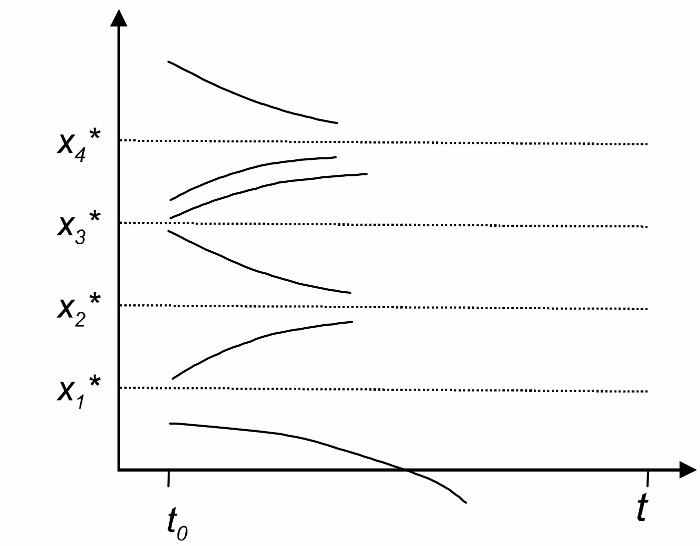


Figure 4.2.4 *Qualitative solutions of an ODE for different initial conditions II*

If an economy can be represented by such a differential equation $\dot{x}(t) = g(x(t))$, one would call fixpoints long-run equilibria. There are stable equilibria and unstable equilibria and it depends on the underlying system (the assumptions that implied the differential equation $\dot{x} = g(x)$) which equilibrium would be considered to be the economically relevant one.

As in the system with one fixpoint, we can qualitatively plot solutions of $x(t)$ over time, given different initial values for $x(0)$. This is shown in fig. 4.2.4 which again highlights the stability properties of fixpoints x_1^* to x_4^*.

4.2.2 Two-dimensional systems I - An example

We now extend our qualitative analysis of differential equations to two-dimensional systems. This latter case allows for an analysis of more complex systems than simple one-dimensional differential equations. In almost all economic models with optimal saving decisions, a reduced form consisting of at least two differential equations will result. We start here with an example before we analyse two-dimensional systems more generally in the next chapter.

- The system

Consider the following differential equation system,

$$\dot{x}_1 = x_1^\alpha - x_2, \quad \dot{x}_2 = b + x_1^{-1} - x_2, \quad 0 < \alpha < 1 < b.$$

Assume that for economic reasons we are interested in properties for $x_i > 0$.

- Fixpoint

The first question is as always whether there is a fixpoint at all. In a two-dimensional system, a fixpoint $x^* = (x_1^*, x_2^*)$ is two-dimensional as well. The fixpoint is defined such that both variables do not change over time, i.e. $\dot{x}_1 = \dot{x}_2 = 0$. If such a point exists, it must satisfy

$$\dot{x}_1 = \dot{x}_2 = 0 \Leftrightarrow (x_1^*)^\alpha = x_2^*, \quad x_2^* = b + (x_1^*)^{-1}.$$

By inserting the second equation into the first, x_1^* is determined by $(x_1^*)^\alpha = b + (x_1^*)^{-1}$ and x_2^* follows from $x_2^* = (x_1^*)^\alpha$. Analyzing the properties of the equation $(x_1^*)^\alpha = b + (x_1^*)^{-1}$ would then show that x_1^* is unique: The left-hand side increases monotonically from 0 to infinity for $x_1^* \in [0, \infty[$ while the right-hand side decreases monotonically from infinity to b. Hence, there must be an intersection point and there can be only one as functions are monotonic. As x_1^* is unique, so is $x_2^* = (x_1^*)^\alpha$.

• Zero-motion lines and pairs of arrows

Having derived the fixpoint, we now need to understand the behaviour of the system more generally. What happens to x_1 and x_2 when $(x_1, x_2) \neq (x_1^*, x_2^*)$? To answer this question, the concept of zero-motion lines is very useful. A zero-motion line is a line for a variable x_i which marks the points for which the variable x_i does not change, i.e. $\dot{x}_i = 0$. For our two-dimensional differential equation system, we obtain two zero-motion lines,

$$\begin{aligned} \dot{x}_1 \geq 0 \Leftrightarrow x_2 \leq x_1^\alpha, \\ \dot{x}_2 \geq 0 \Leftrightarrow x_2 \leq b + x_1^{-1}. \end{aligned} \tag{4.2.1}$$

In addition to the equality sign, we also analyse here at the same time for which values x_i rises. Why this is useful will soon become clear. We can now plot the curves where $\dot{x}_i = 0$ in a diagram. In contrast to the one-dimensional graphs in the previous chapter, we now have the variables x_1 and x_2 on the axes (and not the change of one variable on the vertical axis). The intersection point of the two zero-motion lines gives the fix point $x^* = (x_1^*, x_2^*)$ which we derived analytically above.

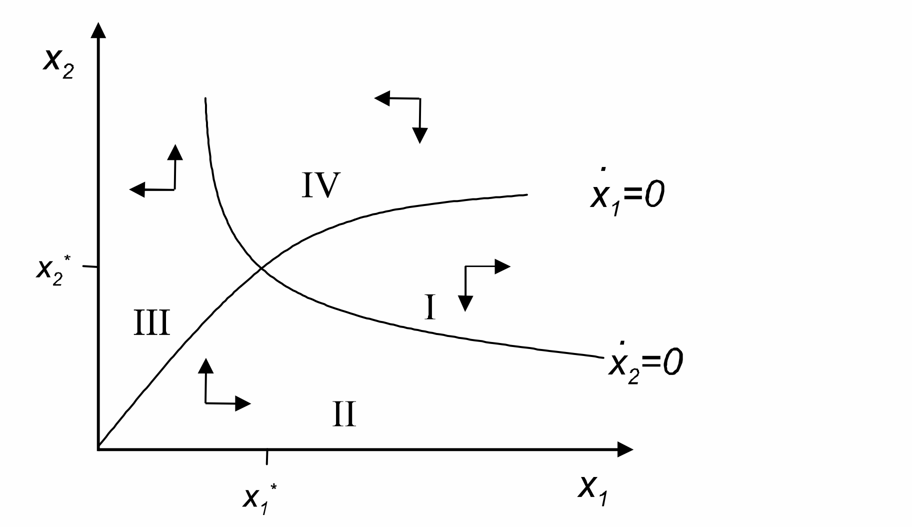


Figure 4.2.5 *First steps towards a phase diagram*

In addition to showing where variables do not change, the zero-motion lines also delimit regions where variables do change. Looking at (4.2.1) again shows (why we used the $\geq$ and not the $=$ sign and) that the variable x_1 increases whenever $x_2 < x_1^\alpha$. Similarly, the variable x_2 increases, whenever $x_2 < b + x_1^{-1}$. The directions in which variables change can then be plotted into this diagram by using arrows. In this diagram, there is a pair of arrows per region as two directions (one for x_1, one for x_2) need to be indicated. This is in principle identical to the arrows we used in the analysis of the one-dimensional systems. If the system finds itself in one of these four regions, we know qualitatively, how variables change over time: Variables move to the south-east in region I, to the north-east in region II, to the north-west in region III and to the south-west in region IV.

- Trajectories

Given the zero-motion lines, the fixpoint and the pairs of arrows, we are now able to draw trajectories into this phase diagram. We will do so and analyse the implications of pairs of arrows further once we have generalized the derivation of a phase diagram.

4.2.3 Two-dimensional systems II - The general case

After this specific example, we will now look at a more general differential equation system and will analyse it by using a phase diagram.

- The system

Consider two differential equations where functions $f(.)$ and $g(.)$ are continuous and differentiable,

$$\dot{x}_1 = f(x_1, x_2), \quad \dot{x}_2 = g(x_1, x_2). \tag{4.2.2}$$

For the following analysis, we will need four assumptions on partial derivatives; all of them are positive apart from $f_{x_1}(.)$,

$$f_{x_1}(.) < 0, \quad f_{x_2}(.) > 0, \quad g_{x1}(.) > 0, \quad g_{x_2}(.) > 0. \tag{4.2.3}$$

Note that, provided we are willing to make the assumptions required by the theorems in ch. 4.1.2, we know that there is a unique solution to this differential equation system, i.e. $x_1(t)$ and $x_2(t)$ are unambiguously determined given two boundary conditions.

- Fixpoint

The first question to be tackled is whether there is an equilibrium at all. Is there a fixpoint x^* such that $\dot{x}_1 = \dot{x}_2 = 0$? To this end, set $f(x_1, x_2) = 0$ and $g(x_1, x_2) = 0$ and plot the implicitly defined functions in a graph.

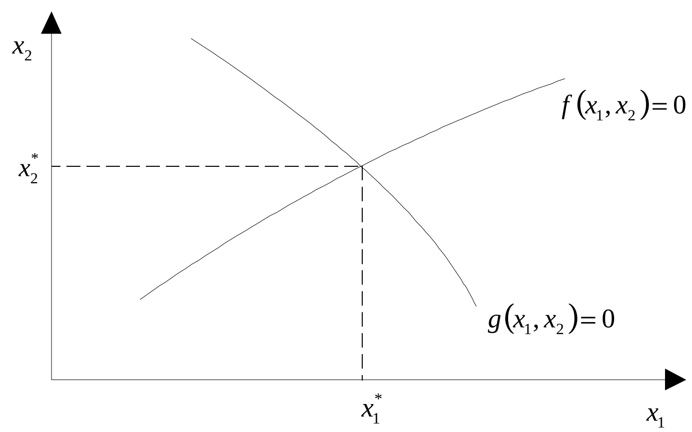


Figure 4.2.6 *Zero motion lines with a unique steady state*

By the implicit function theorem - see (2.3.3) - and the assumptions made in (4.2.3), one zero motion line is increasing and one is decreasing. If we are further willing to assume that functions are not monotonically approaching an upper and lower bound, we know that there is a unique fixpoint $(x_1^*, x_2^*) \equiv x^*$.

- General evolution

Now we ask again what happens if the state of the system differs from x^*, i.e. if either x_1 or x_2 or both differ from their steady state values. To find an answer, we have to determine the sign of $f(x_1, x_2)$ and $g(x_{1,}x_2)$ for some (x_1, x_2). Given (4.2.2), x_1 would increase for a positive $f(.)$ and x_2 would increase for a positive $g(.)$. For any known functions $f(x_1, x_2)$ and $g(x_{1,}x_2)$, one can simply plot a 3-dimensional figure with x_1 and x_2 on the axes in the plane and with time derivatives on the vertical axis.

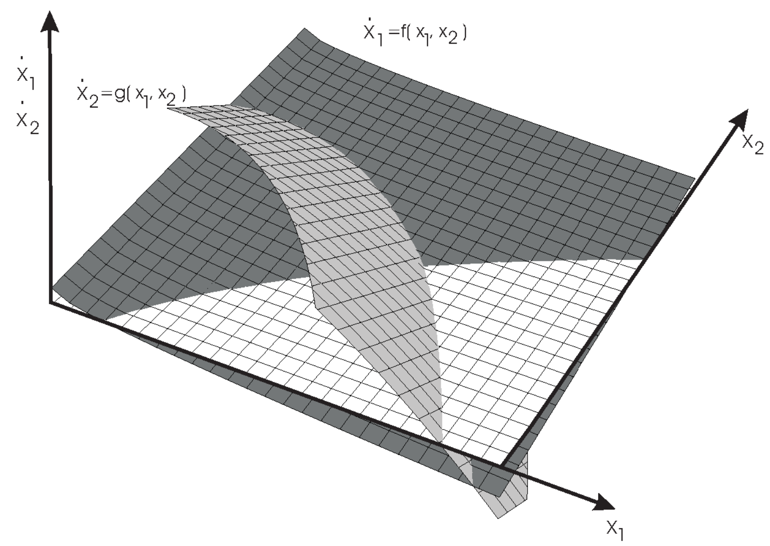


Figure 4.2.7 *A three-dimensional illustration of two differential equations and their zero-motion lines*

The white area in this figure is the horizontal plane, i.e. where $\dot{x}_1$ and $\dot{x}_2$ are zero. The dark surface illustrates the law of motion for x_1 as does the grey surface for x_2. The intersection of the dark surface with the horizontal plane gives the loci on which x_1 does not change. The same is true for the grey surface and x_2. Clearly, at the intersection point of these zero-motion lines we find the steady state x^*.

When working with two-dimensional figures and without the aid of computers, we start from the zero-motion line for, say, x_1 and plot it into a "normal" figure.

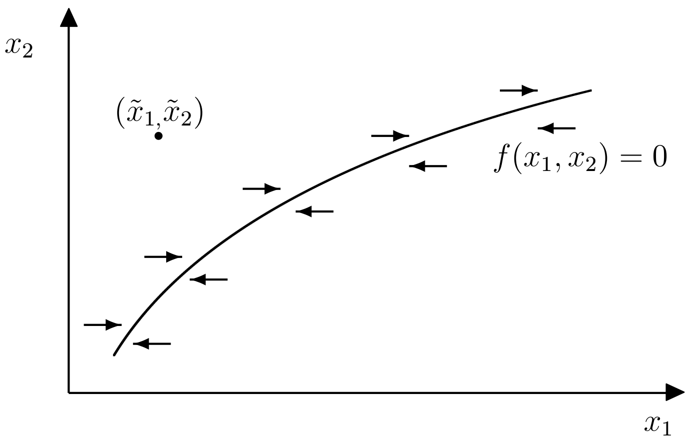


Figure 4.2.8 *Step 1 in constructing a phase diagram*

Now consider a point $(\tilde{x}_1, \tilde{x}_2)$. As we know that $\dot{x}_1 = 0$ on $f(x_1, x_2) = 0$ and that from (4.2.3) $f_{x_2}(.) > 0$, we know that moving from $\tilde{x}_1$ on the zero-motion line vertically to $(\tilde{x}_1, \tilde{x}_2)$, $f(.)$ is increasing. Hence, x_1 is increasing, $\dot{x}_1 > 0$, at $(\tilde{x}_1, \tilde{x}_2)$. As this line of reasoning holds for any $(\tilde{x}_1, \tilde{x}_2)$ above the zero-motion line, x_1 is increasing everywhere above $f(x_1, x_2) = 0$. As a consequence, x_1 is decreasing everywhere below the zero-motion line. This movement is indicated by the arrows in the above figure.

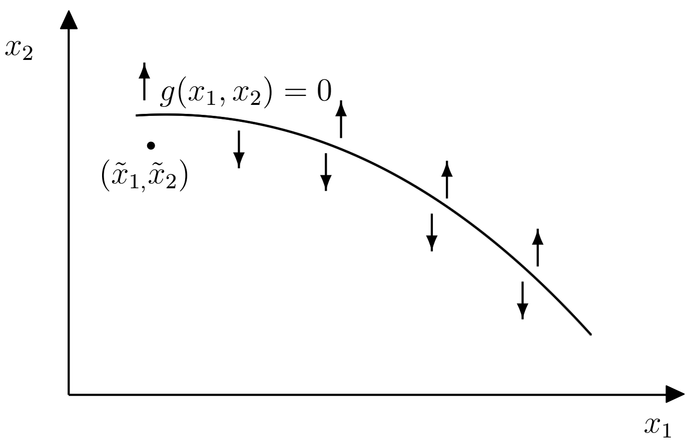


Figure 4.2.9 *Step 2 in constructing a phase diagram*

Let us now consider the second zero-motion line, $\dot{x}_2 = 0 \Leftrightarrow g(x_1, x_2) = 0$, and look again at the point $(\tilde{x}_1, \tilde{x}_2)$. When we start from the point $\tilde{x}_2$ on the zero motion line and move towards $(\tilde{x}_1, \tilde{x}_2)$, $g(x_1, x_2)$ is decreasing, given the derivative $g_{x1}(.) > 0$ from (4.2.3). Hence, for any points to the left of $g(x_1, x_2) = 0$, x_2 is decreasing. Again, this is shown in the above figure.

- Fixpoints and trajectories

We can now represent the directions in which x_1 and x_2 are moving into a single phase-diagram by plotting one arrow each into each of the four regions limited by the zero-motion lines. Given that the arrows can either indicate an increase or a decrease for both x_1 and x_2, there are two times two different combinations of arrows, i.e. four regions. When we add some representative trajectories, a complete phase diagram results.

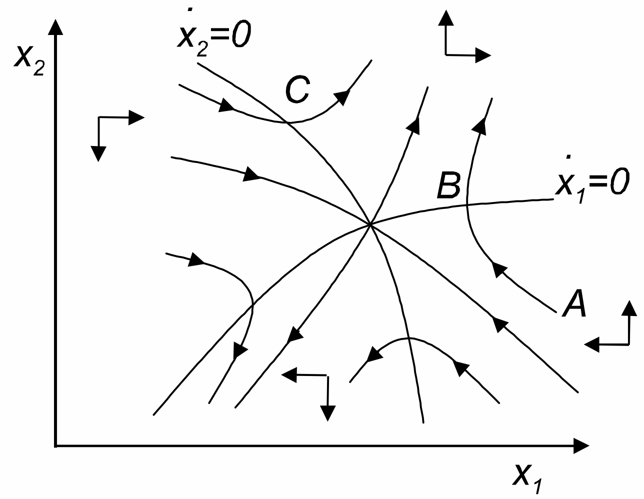


Figure 4.2.10 *Phase diagram for a saddle point*

Adding trajectories is relatively easy when paying attention to the zero-motion lines. When trajectories are plotted "far away" from zero-motion lines, the arrow pairs indicate whether the movement is towards the "north-east", the "south-east" or the other two possible directions. At point A in the phase diagram for a saddle point, the movement is towards the "north-west". Note that the arrows represent first derivatives only. As second derivatives are not taken into account (usually), we do not know whether the trajectory moves more and more to the north or more and more to the west. The arrow-pairs are also consistent with wave-like trajectories, as long as the movement is always towards the north-west.

Precise information on the shape of the trajectories is available when we look at the points where trajectories cross the zero-motion lines. On a zero-motion line, the variable to which this zero-motion line belongs does not change. Hence all trajectories cross zero-motion lines either vertically or horizontally. When we look at point B above, the trajectory moves from the north-west to the north-east region. The variable x_1 changes direction and begins to rise after having crossed its zero-motion line vertically.

An example where a zero-motion line is crossed horizontally is point C. To the left of point C, x_1 rises and x_2 falls. On the zero-motion line for x_2, x_2 does not change but x_1 continues to rise. To the right of C, both x_1 and x_2 increase. Similar changes in direction can be observed at other intersection points of trajectories with zero-motion lines.

4.2.4 Types of phase diagrams and fixpoints

- Types of fixpoints

As has become clear by now, the partial derivatives in (4.2.3) are crucial for the slope of the zero-motion lines and for the direction of movements of variables x_1 and x_2. Depending on the signs of the partial derivatives, various phase diagrams can occur. As there are two possible directions for each variable, these phase diagrams can be classified into four typical groups, depending on the properties of their fixpoint.

Definition 4.2.2 *A fixpoint is called a*

$$\left.\begin{array}{c}\text{center}\\ \text{saddle point}\\ \text{focus}\\ \text{node}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{zero}\\ \text{two}\\ \text{all}\\ \text{all}\end{array}\right\} \text{trajectories pass through the fixpoint}$$

$$\text{and} \left\{\begin{array}{l} \\ \\ \text{on} \left\{\begin{array}{c}\text{at least one trajectory, both variables are non-monotonic}\\ \text{all trajectories, one or both variables are monotonic}\end{array}\right.\end{array}\right.$$

A node and a focus can be either stable or unstable.

- Illustration

Here is now an overview of some typical phase diagrams.

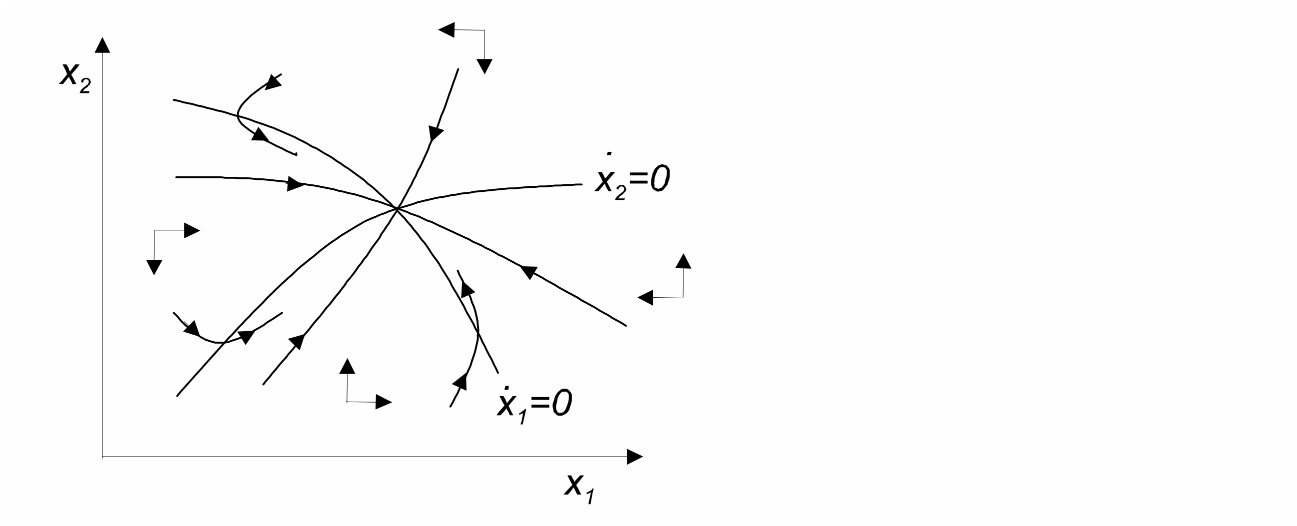


Figure 4.2.11 *Phase diagram for a node*

This first phase diagram shows a node. A node is a fixpoint through which all trajectories go and where the time paths implied by trajectories are monotonic for at least one variable. As drawn here, it is a stable node, i.e. for any initial conditions, the system ends up in the fixpoint. An unstable node is a fixpoint from which all trajectories start. A phase diagram for an unstable node would look like the one above but with all directions of motions reversed.

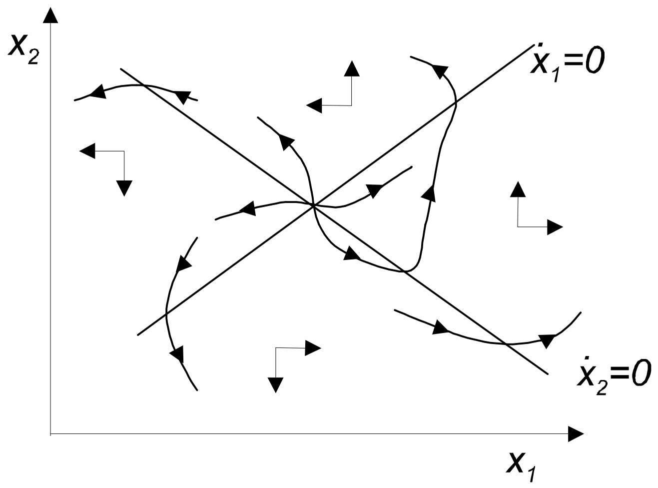


Figure 4.2.12 *Phase diagram for a focus*

A phase diagram with a focus looks similar to one with a node. The difference lies in the non-monotonic paths of the trajectories. As drawn here, x_1 or x_2 first increase and then decrease on some trajectories.

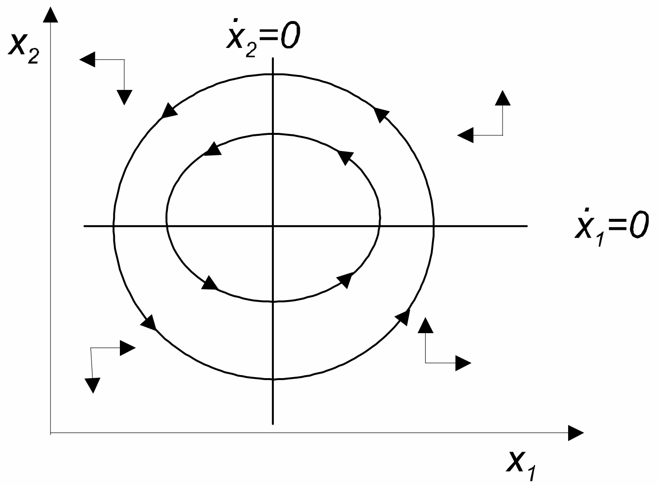


Figure 4.2.13 *Phase diagram for a center*

A circle is a very special case for a differential equation system. It is rarely found in models with optimizing agents. The standard example is the predator-prey model, $\dot{x} = \alpha x - \beta xy$, $\dot{y} = -\gamma y + \delta xy$, where α, β, γ, and δ are positive constants. This is also called the Lotka-Volterra model. No closed-form solution has been found so far.

- Limitations

It should be noted that a phase diagram analysis allows us to identify a saddle point. If no saddle point can be identified, it is generally not possible to distinguish between a node, focus or center. In the linear case, more can be deduced from a graphical analysis. This is generally not necessary, however, as there is a closed-form solution. The definition of various types of fixpoints is then based on Eigenvalues of the system. See ch. 4.5.

4.2.5 Multidimensional systems

If we have higher-dimensional problems where $x \in R^n$ and $n > 2$, phase diagrams are obviously difficult to draw. In the three-dimensional case, plotting zero motion surfaces sometimes helps to gain some intuition. A graphical solution will generally, however, not allow us to identify equilibrium properties like saddle-path or saddle-plane behaviour.

4.3 Linear differential equations

This section will focus on a special case of the general ODE defined in (4.1.1). The special aspect consists of making the function $f(.)$ in (4.1.1) linear in $x(t)$. By doing this, we obtain a linear differential equation,

$$\dot{x}(t) = a(t)\,x(t) + b(t), \tag{4.3.1}$$

where $a(t)$ and $b(t)$ are functions of time. This is the most general case for a one dimensional linear differential equation.

4.3.1 Rules on derivatives

Before analyzing (4.3.1) in more detail, we first need some other results which will be useful later. This section therefore first presents some rules on how to compute derivatives. It is by no means intended to be comprehensive or go in any depth. It presents rules which have shown by experience to be of importance.

- Integrals

Definition 4.3.1 *A function $F(x) \equiv \int f(x)\,dx$ is the indefinite integral of a function $f(x)$ if*

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int f(x)\,dx = f(x) \tag{4.3.2}$$

This definition implies that there are infinitely many integrals of $f(x)$. If $F(x)$ is an integral, then $F(x)+c$, where c is a constant, is an integral as well.

- Leibniz rule

We present here a rule for computing the derivative of an integral function. Let there be a function $z(x)$ with argument x, defined by the integral

$$z(x) \equiv \int_{a(x)}^{b(x)} f(x,y)\,dy,$$

where $a(x)$, $b(x)$ and $f(x,y)$ are differentiable functions. Note that x is the only argument of z, as y is integrated out on the right-hand side. Then, the Leibniz rule says that the derivative of this function with respect to x is

$$\frac{d}{dx}z(x) = b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y)\,dy. \tag{4.3.3}$$

The following figure illustrates this rule.

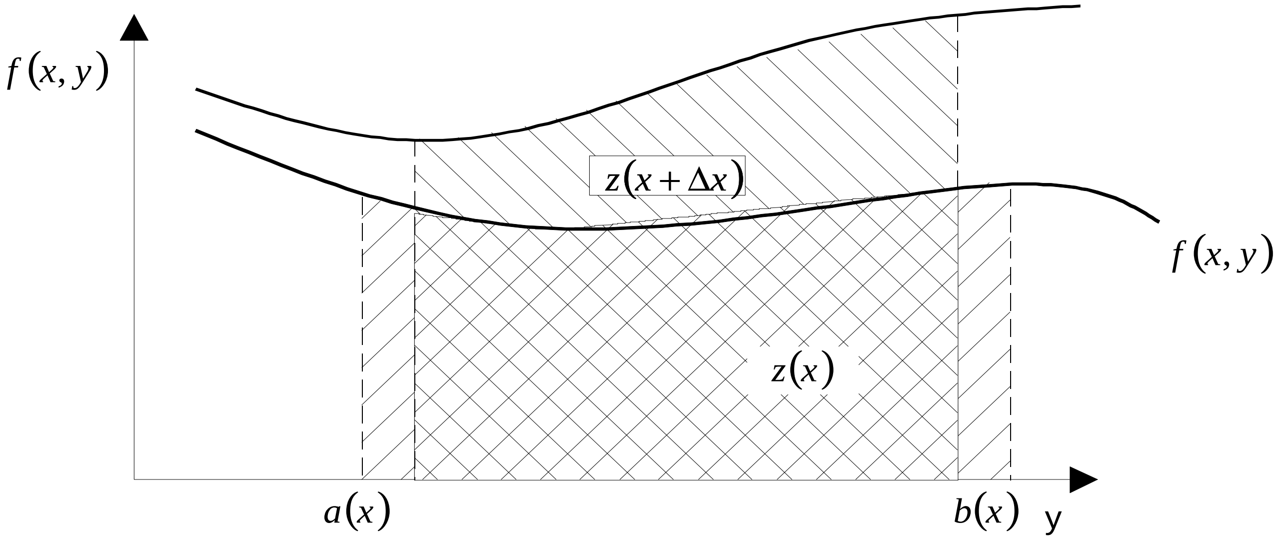


Figure 4.3.1 *Illustration of the differentiation rule*

Let x increase by a small amount. Then the integral changes at three margins: The upper bound, the lower bound and the function $f(x,y)$ itself. As drawn here, the upper bound $b(x)$ and the function $f(x,y)$ increase and the lower bound $a(x)$ decreases in x. As a consequence, the area below the function $f(.)$ between bounds $a(.)$ and $b(.)$ increases because of three changes: the increase to the left because of $a(.)$, the increase to the right because of $b(.)$ and the increase upwards because of $f(.)$ itself. Clearly, this figure changes when the derivatives of $a(.)$, $b(.)$ and $f(.)$ with respect to x have a different sign than the ones assumed here.

- Another derivative with an integral

Now consider a function of the type $y = \int_a^b f(x(i))\, di$. Functions of this type will be encountered frequently as objective functions, e.g. intertemporal utility or profit functions. What is the derivative of y with respect to $x(i)$? It is given by $\partial y / \partial x(i) = f'(x(i))$. The integral is not part of this derivative as the derivative is computed for one specific $x(i)$ and not for all $x(i)$ with i lying between a and b. Note the analogy to maximizing a sum as e.g. in (3.1.4). The integration variable i here corresponds to the summation index τ in (3.1.4). When one specific point i is chosen (say $i = (a+b)/2$), all derivatives of the other $f(x(i))$ with respect to this specific $x(i)$ are zero.

- Integration by parts (for indefinite and definite integrals)

Proposition 4.3.1 *For two differentiable functions $u(x)$ and $v(x)$,*

$$\int u'(x)\, v(x)\, dx = u(x)\, v(x) - \int u(x)\, v'(x)\, dx. \tag{4.3.4}$$

Proof. We start by observing that

$$(u(x)\, v(x))' = u'(x)\, v(x) + u(x)\, v'(x),$$

where we used the product rule. Integrating both sides by applying $\int \cdot\, dx$, gives

$$u(x)\, v(x) = \int u'(x)\, v(x)\, dx + \int u(x)\, v'(x)\, dx.$$

Rearranging gives (4.3.4). ■

Equivalently, one can show (see the exercise 8) that

$$\int_a^b \dot{x} y dt = [xy]_a^b - \int_a^b x \dot{y} dt. \tag{4.3.5}$$

4.3.2 Forward and backward solutions of a linear differential equation

We now return to our linear differential equation $\dot{x}(t) = a(t)\, x(t) + b(t)$ from (4.3.1). It will now be solved. Generally speaking, a solution to a differential equation is a function $x(t)$ which satisfies this equation. A solution can be called a time path of x when t represents time.

- General solution of the non-homogeneous equation

The differential equation in (4.3.1) has, as all differential equations, an infinite number of solutions. The general solution reads

$$x(t) = e^{\int a(t)dt}\left(\tilde{x} + \int e^{-\int a(t)dt} b(t)\,dt\right). \tag{4.3.6}$$

Here, $\tilde{x}$ is some arbitrary constant. As this constant is arbitrary, (4.3.6) indeed provides an infinite number of solutions to (4.3.1).

To see that (4.3.6) is a solution to (4.3.1) indeed, remember the definition of what a solution is. A solution is a time path $x(t)$ which satisfies (4.3.1). Hence, we simply need to insert the time path given by (4.3.6) into (4.3.1) and check whether (4.3.1) then holds. To this end, compute the time derivative of $x(t)$,

$$\frac{d}{dt}x(t) = e^{\int a(t)dt} a(t)\left(\tilde{x} + \int e^{-\int a(t)dt} b(t)\,dt\right) + e^{\int a(t)dt} e^{-\int a(t)dt} b(t),$$

where we have used the definition of the integral in (4.3.2), $\frac{d}{dx}\int f(x)\,dx = f(x)$. Note that we do not have to apply the product or chain rule since, again by (4.3.2), $\frac{d}{dx}\int g(x)\,h(x)\,dx = g(x)\,h(x)$. Inserting (4.3.6) gives $\dot{x}(t) = a(t)\,x(t) + b(t)$. Hence, (4.3.6) is a solution to (4.3.1).

- Determining the constant $\tilde{x}$

To obtain one particular solution, some value $x(t_0)$ at some point in time t_0 has to be fixed. Depending on whether t_0 lies in the future (where t_0 is usually denoted by T) or in the past, $t < T$ or $t_0 < t$, the equation is solved forward or backward.

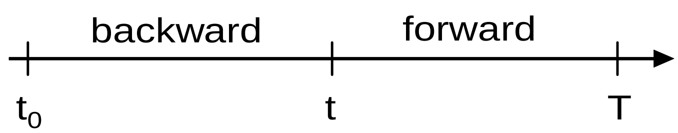


Figure 4.3.2 *Illustrating backward solution (initial value problem) and forward solution (boundary value problem)*

We start with the backward solution, i.e. where $t_0 < t$. Let the initial condition be $x(t_0) = x_0$. Then the solution of (4.3.1) is

$$\begin{aligned} x(t) &= e^{\int_{t_0}^{t} a(\tau)d\tau}\left[x_0 + \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} a(u)du} b(\tau)\,d\tau\right] \\ &= x_0 e^{\int_{t_0}^{t} a(\tau)d\tau} + \int_{t_0}^{t} e^{\int_{\tau}^{t} a(u)du} b(\tau)\,d\tau. \end{aligned} \tag{4.3.7}$$

Some intuition for this solution can be gained by considering special cases. Look first at the case where $b(t) = 0$ for all t (and therefore all τ). The variable $x(t)$ then grows at a variable growth rate $a(t)$, $\dot{x}(t)/x(t) = a(t)$ from (4.3.1). The solution to this ODE is

$$x(t) = x_0 e^{\int_{t_0}^{t} a(\tau)d\tau} \equiv x_0 e^{\bar{a}[t-t_0]}$$

where $\bar{a} \equiv \frac{\int_{t_0}^{t} a(\tau)d\tau}{t-t_0}$ is the average growth rate of a between t_0 and t. The solution $x_0 e^{\bar{a}[t-t_0]}$ has the same structure as the solution for a constant a - this ODE implies an exponential increase of $x(t)$. Looking at $\bar{a}$ then shows that this exponential increase now takes place at the average of $a(t)$ over the period we are looking at.

Now allow for a positive $b(t)$. The solution (4.3.7) says that when a $b(\tau)$ is added in τ, the effect on $x(t)$ is given by the initial $b(\tau)$ times an exponential increase factor $e^{\int_{\tau}^{t} a(u)du}$ that takes the increase from τ to t into account. As a $b(\tau)$ is added at each τ, the outer integral $\int_{t_0}^{t} .d\tau$ "sums" over all these individual contributions.

The forward solution is required if t_0, which we rename T for ease of distinction, lies in the future of t, $T > t$. With a terminal condition $x(T) = x_T$, the solution then reads

$$x(t) = x_T e^{-\int_{t}^{T} a(\tau)d\tau} - \int_{t}^{T} e^{-\int_{t}^{\tau} a(u)du} b(\tau)\, d\tau. \tag{4.3.8}$$

A similar intuitive explanation as after (4.3.7) can be given for this equation.

- Verification

We now show that (4.3.7) and (4.3.8) are indeed a solution for (4.3.1). Using the Leibniz rule from (4.3.3), the time derivative of (4.3.7) is given by

$$\dot{x}(t) = e^{\int_{t_0}^{t} a(\tau)d\tau} a(t)\, x_0 + e^{\int_{t}^{t} a(u)du} b(t) + \int_{t_0}^{t} e^{\int_{\tau}^{t} a(u)du} a(t)\, b(\tau)\, d\tau.$$

When we pull out $a(t)$ and reinsert (4.3.7), we find

$$\begin{aligned}\dot{x}(t) &= a(t) \left(e^{\int_{t_0}^{t} a(\tau)d\tau} x_0 + \int_{t_0}^{t} e^{\int_{\tau}^{t} a(u)du} b(\tau)\, d\tau \right) + b(t) \\ &= a(t)\, x(t) + b(t).\end{aligned}$$

This shows us that our function $x(t)$ in (4.3.7) is in fact a solution of (4.3.1) as $x(t)$ in (4.3.7) satisfies (4.3.1).

The time derivative for the forward solution in (4.3.8) is

$$\begin{aligned}\dot{x}(t) &= e^{-\int_{t}^{T} a(\tau)d\tau} a(t)\, x_T + b(t) - \int_{t}^{T} e^{-\int_{t}^{\tau} a(u)du} b(\tau)\, d\tau\, a(t) \\ &= a(t) \left(e^{-\int_{t}^{T} a(\tau)d\tau} x_T - \int_{t}^{T} e^{-\int_{t}^{\tau} a(u)du} b(\tau)\, d\tau \right) + b(t) \\ &= a(t)\, x(t) + b(t).\end{aligned}$$

Here, (4.3.8) was also reinserted into the second step. This shows that (4.3.8) is also a solution of (4.3.1).

4.3.3 Differential equations as integral equations

Any differential equation can be written as an integral equation. While we will work with the "usual" differential equation representation most of the time, we introduce the integral representation here as it will be used frequently later when computing moments in stochastic setups. Understanding the integral version of differential equations in this deterministic setup allows for an easier understanding of integral representations of stochastic differential equations later.

- The principle

The non-autonomous differential equation $\dot{x} = f(t, x)$ can be written equivalently as an integral equation. To this end, write this equation as $dx = f(t, x)\, dt$ or, after substituting s for t, as $dx = f(s, x)\, ds$. Now apply the integral $\int_0^t$on both sides. This gives the integral version of the differential equation $\dot{x} = f(t, x)$ which reads

$$\int_0^t dx = x(t) - x(0) = \int_0^t f(s, x)\, ds.$$

- An example

The differential equation $\dot{x} = a(t)\, x$ is equivalent to

$$x(t) = x_0 + \int_0^t a(s)\, x(s)\, ds. \tag{4.3.9}$$

Computing the derivative of this equation with respect to time t gives, using (4.3.3), $\dot{x}(t) = a(t)\, x(t)$ again.

The presence of an integral in (4.3.9) should not lead one to confuse (4.3.9) with a solution of $\dot{x} = a(t)\, x$ in the sense of the last section. Such a solution would read $x(t) = x_0 e^{\int_0^t a(s) ds}$.

4.4 Examples

4.4.1 Backward solution: A growth model

Consider an example inspired by growth theory. Let the capital stock of the economy follow $\dot{K} = I - \delta K$, gross investment minus depreciation gives the net increase of the capital stock. Let gross investment be determined by a constant saving rate times output, $I = sY(K)$ and the technology be given by a linear AK specification. The complete differential equation then reads

$$\dot{K} = sAK - \delta K = (sA - \delta)\, K.$$

Its solution is (see ch. 4.3.2) $K(t) = \gamma e^{(sA-\delta)t}$. As in the qualitative analysis above, we found a multitude of solutions, depending on the constant γ. If we specify an initial condition, say we know the capital stock at $t = 0$, i.e. $K(0) = K_0$, then we can fix the constant γ by $\gamma = K_0$ and our solution finally reads $K(t) = K_0 e^{(sA-\delta)t}$.

4.4.2 Forward solution: Budget constraints

As an application of differential equations, consider the budget constraint of a household. As in discrete time in ch. 3.5.2, budget constraints can be expressed in a dynamic and an intertemporal way. We first show here how to derive the dynamic version and then how to obtain the intertemporal version from solving the dynamic one.

- Deriving a nominal dynamic budget constraint

Following the idea of ch. 2.5.5, let us first derive the dynamic budget constraint. In contrast to ch. 2.5.5 in discrete time, we will see how straightforward a derivation is in continuous time.

We start from the definition of nominal wealth. We have only one asset here. Nominal wealth is therefore given by $a = kv$, where k is the household's physical capital stock and v is the value of one unit of the capital stock. One can alternatively think of k as the number of shares held by the household. By computing the time derivative, wealth of a household changes according to

$$\dot{a} = \dot{k}v + k\dot{v}. \tag{4.4.1}$$

If the household wants to save, it can buy capital goods. The household's nominal savings in t are given by $s = w^K k + w - pc$, the difference between factor rewards for capital (value marginal product times capital owned), labour income and expenditure. Dividing savings by the value of a capital good, i.e. the price of a share, gives the number of shares bought (or sold if savings are negative),

$$\dot{k} = \frac{w^K k + w - pc}{v}. \tag{4.4.2}$$

Inserting this into (4.4.1), the equation for wealth accumulation gives, after reintroducing wealth a by replacing k by a/v,

$$\dot{a} = w^K k + w - pc + k\dot{v} = \left(w^K + \dot{v}\right) k + w - pc = \frac{w^K + \dot{v}}{v} a + w - pc.$$

Defining the nominal interest rate as

$$i \equiv \frac{w^K + \dot{v}}{v}, \tag{4.4.3}$$

we have the nominal budget constraint

$$\dot{a} = ia + w - pc. \tag{4.4.4}$$

This shows why it is wise to always derive a budget constraint. Without a derivation, (4.4.3) is missed out and the meaning of the interest rate i in the budget constraint is not known.

- Finding the intertemporal budget constraint

We can now obtain the intertemporal budget constraint from solving the dynamic one in (4.4.4). Using the forward solution from (4.3.8), we take $a(T) = a_T$ as the terminal condition lying with $T > t$ in the future. We are in t today. The solution is then

$$a(t) = e^{-\int_t^T i(\tau)d\tau} a_T - \int_t^T e^{-\int_t^\tau i(u)du} [w(\tau) - p(\tau) c(\tau)] d\tau \Leftrightarrow$$

$$\int_t^T D(\tau) w(\tau) d\tau + a(t) = D(T) a_T + \int_t^T D(\tau) p(\tau) c(\tau) d\tau,$$

where $D(\tau) \equiv e^{-\int_t^\tau i(u)du}$ defines the discount factor. As we have used the forward solution, we have obtained an expression which easily lends itself to an economic interpretation. Think of an individual who - at the end of his life - does not want to leave any bequest, i.e. $a_T = 0$. Then, this intertemporal budget constraint requires that current wealth on the left-hand side, consisting of the present value of life-time labour income plus financial wealth $a(t)$ needs to equal the present value of current and future expenditure on the right-hand side.

Now imagine that $a_T > 0$ as the individual does plan to leave a bequest. Then this bequest is visible as an expenditure on the right-hand side. Current wealth plus the present value of wage income on the left must then be high enough to provide for the present value $D(T) a_T$ of this bequest and the present value of consumption.

What about debt in T? Imagine there is a fairy godmother who pays all debts left at the end of a life. With $a_T < 0$ the household can consume more than current wealth $a(t)$ and the present value of labour income - the difference is just the present value of debt, $D(T) a_T < 0$.

Now let the future point T in time go to infinity. Expressing $\lim_{T\to\infty} \int_t^T f(\tau) d\tau$ as $\int_t^\infty f(\tau) d\tau$, the budget constraint becomes

$$\int_t^\infty D(\tau) w(\tau) d\tau + a(t) = \lim_{T\to\infty} D(T) a_T + \int_t^\infty D(\tau) p(\tau) c(\tau) d\tau$$

What would a negative present value of future wealth now mean, i.e. $\lim_{T\to\infty} D(T) a_T < 0$? If there was such a fairy godmother, having debt would allow the agent to permanently consume above its income levels and pay for this difference by accumulating debt. As fairy godmothers rarely exist in real life - especially when we think about economic aspects - economists usually assume that

$$\lim_{T\to\infty} D(T) a_T = 0. \tag{4.4.5}$$

This condition is often called solvency or no-Ponzi game condition. Note that the no-Ponzi game condition is a different concept from the boundedness condition in ch. 5.3.2 or the transversality condition in ch. 5.4.3.

- Real wealth

We can also start from the definition of the household's real wealth, measured in units of the consumption good, whose price is p. Real wealth is then $a^r = \frac{kv}{p}$. The change in real wealth over time is then (apply the log on both sides and derive with respect to time),

$$\frac{\dot{a}^r}{a^r} = \frac{\dot{k}}{k} + \frac{\dot{v}}{v} - \frac{\dot{p}}{p}.$$

Inserting the increase into the capital stock obtained from (4.4.2) gives

$$\frac{\dot{a}^r}{a^r} = \frac{w^K k + w - pc}{vk} + \frac{\dot{v}}{v} - \frac{\dot{p}}{p} = \frac{\left(w^K k + w - pc\right) p^{-1}}{vkp^{-1}} + \frac{\dot{v}}{v} - \frac{\dot{p}}{p}.$$

Using the expression for real wealth a^r,

$$\begin{aligned} \dot{a}^r &= w^K \frac{k}{p} + \frac{w}{p} - c + \frac{\dot{v}}{v} a^r - \frac{\dot{p}}{p} a^r = \frac{w^K}{v} a^r + \frac{w}{p} - c + \frac{\dot{v}}{v} a^r - \frac{\dot{p}}{p} a^r \\ &= \left(\frac{w^K + \dot{v}}{v} - \frac{\dot{p}}{p} \right) a^r + \frac{w}{p} - c = r a^r + \frac{w}{p} - c. \end{aligned} \tag{4.4.6}$$

Hence, the real interest rate r is - by definition -

$$r = \frac{w^K + \dot{v}}{v} - \frac{\dot{p}}{p}.$$

The difference to (4.4.3) simply lies in the inflation rate: Nominal interest rate minus inflation rate gives real interest rate. Solving the differential equation (4.4.6) again provides the intertemporal budget constraint as in the nominal case above.

Now assume that the price of the capital good equals the price of the consumption good, $v = p$. This is the case in an economy where there is one homogeneous output good as in (2.4.9) or in (9.3.4). Then, the real interest rate is equal to the marginal product of capital, $r = w^K/p$.

4.4.3 Forward solution again: capital markets and utility

- The capital market no-arbitrage condition

Imagine you own wealth of worth $v(t)$. You can invest it on a bank account which pays a certain return $r(t)$ per unit of time or you can buy shares of a firm which cost $v(t)$ and which yield dividend payments $\pi(t)$ and are subject to changes $\dot{v}(t)$ in its worth. In a world of perfect information and assuming that in some equilibrium agents hold both assets, the two assets must yield identical income streams,

$$r(t)\, v(t) = \pi(t) + \dot{v}(t). \tag{4.4.7}$$

This is a linear differential equation in $v(t)$. As just motivated, it can be considered as a no-arbitrage condition. Note, however, that it is structurally equivalent to (4.4.3), i.e. this no-arbitrage condition can be seen to just define the interest rate $r(t)$.

Whatever the interpretation of this differential equation is, solving it forward with a terminal condition $v(T) = v_T$ gives according to (4.3.8)

$$v(t) = e^{-\int_t^T r(\tau)d\tau} v_T + \int_t^T e^{-\int_t^\tau r(u)du} \pi(\tau)\, d\tau.$$

Letting T go to infinity, we have

$$v(t) = \int_t^\infty e^{-\int_t^\tau r(u)du} \pi(\tau)\, d\tau + \lim_{T\to\infty} e^{-\int_t^T r(\tau)d\tau} v_T.$$

This forward solution stresses the economic interpretation of $v(t)$: The value of an asset depends on the future income stream - dividend payments $\pi(\tau)$ - that are generated from owning this asset. Note that it is usually assumed that there are no bubbles, i.e. the limit is zero so that the fundamental value of an asset is given by the first term.

For a constant interest rate and dividend payments and no bubbles, the expression for $v(t)$ simplifies to $v = \pi/r$.

- The utility function

Consider an intertemporal utility function as it is often used in continuous time models,

$$U(t) = \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau))\, d\tau. \tag{4.4.8}$$

This is the standard expression which corresponds to (2.2.12) or (3.1.1) in discrete time. Again, instantaneous utility is given by $u(.)$. It depends here on consumption only, where households consume continuously at each instant τ. Impatience is captured as before by the time preference rate ρ: Higher values attached to present consumption are captured by the discount function $e^{-\rho[\tau-t]}$, whose discrete time analog in (3.1.1) is $\beta^{\tau-t}$.

Using (4.3.3), differentiating with respect to time gives us a linear differential equation,

$$\dot{U}(t) = -u(c(t)) + \int_t^\infty \frac{d}{dt}\left[e^{-\rho[\tau-t]} u(c(\tau))\right] d\tau = -u(c(t)) + \rho U(t).$$

This equation says that overall utility $U(t)$ decreases, as time goes by, by instantaneous consumption $u(c(t))$. When t is over, the opportunity is gone: we can no longer enjoy utility from consumption at t. But $U(t)$ also has an increasing component: as the future comes closer, we gain $\rho U(t)$.

Solving this linear differential equation forward gives

$$U(t) = e^{-\rho[T-t]} U(T) + \int_t^T e^{-\rho[\tau-t]} u(c(\tau))\, d\tau.$$

Letting T go to infinity, we have

$$U(t) = \int_t^T e^{-\rho[\tau-t]} u(c(\tau)) \, d\tau + \lim_{T\to\infty} e^{-\rho[T-t]} U(T).$$

The second term is related to the transversality condition.

4.5 Linear differential equation systems

A differential equation system consists of two or more differential equations which are mutually related to each other. Such a system can be written as

$$\dot{x}(t) = Ax(t) + b,$$

where the vector $x(t)$ is given by $x = (x_1, x_2, x_3, ..., x_n)'$, A is an $n \times n$ matrix with elements a_{ij} and b is a vector $b = (b_1, b_2, b_3, ..., b_n)'$. Note that elements of A and b can be functions of time but not functions of x.

With constant coefficients, such a system can be solved in various ways, e.g. by determining so-called Eigenvalues and Eigenvectors. These systems either result from economic models directly or are the outcome of a linearization of some non-linear system around a steady state. This latter approach plays an important role for local stability analyses (compared to the global analyses we undertook above with phase diagrams). These local stability analyses can be performed for systems of almost arbitrary dimension and are therefore more general and (for the local surrounding of a steady state) more informative than phase diagram analyses.

Please see the references in "further reading" on many textbooks that treat these issues.

4.6 Further reading and exercises

There are many textbooks that treat differential equations and differential equation systems. Any library search tool will provide many hits. This chapter owes insights to Gandolfo (1996) on phase diagram analysis and differential equations and - inter alia - to Brock and Malliaris (1989), Braun (1975) and Chiang (1984) on differential equations. See also Gandolfo (1996) on differential equation systems. The predator-prey model is treated in various biology textbooks. It can also be found on many sites on the Internet. The Leibniz rule was taken from Fichtenholz (1997) and can be found in many other textbooks on differentiation and integration.

The AK specification of a technology was made popular by Rebelo (1991).

Exercises chapter 4
Applied Intertemporal Optimization

Using phase diagrams

1. Phase diagram I
   Consider the following differential equation system,

$$\dot{x}_1 = f(x_1, x_2), \quad \dot{x}_2 = g(x_1, x_2).$$

   Assume

$$f_{x_1}(x_1, x_2) < 0, \quad g_{x_2}(x_1, x_2) < 0, \quad \left.\frac{dx_2}{dx_1}\right|_{f(x_1,x_2)=0} < 0, \quad \left.\frac{dx_2}{dx_1}\right|_{g(x_1,x_2)=0} > 0.$$

   (a) Plot a phase diagram for the positive quadrant.

   (b) What type of fixpoint can be identified with this setup?

2. Phase diagram II

   (a) Plot two phase diagrams for

$$\dot{x} = xy - a, \quad \dot{y} = y - b; a > 0. \tag{4.6.1}$$

   by varying the parameter b.

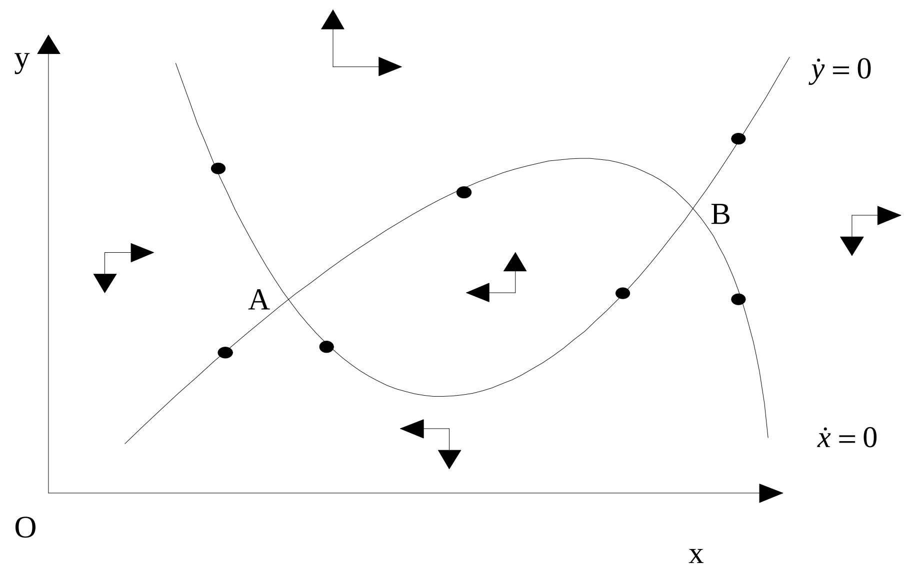
   (b) What type of fixpoints do you find?

   (c) Solve this system analytically. Note that y is linear and can easily be solved. Once this solution is plugged into the differential equation for x, this becomes a linear differential equation as well.

3. Phase diagram III

   (a) Plot paths through points marked by a dot "·" in the figure below.

   (b) What type of fixpoints are A and B?



4. Local stability analysis
   Study local stability properties of the fixpoint of the differential equation system (4.6.1).

5. Phase diagram and fixpoint
   Grossman and Helpman (1991) present a growth model with an increasing number of varieties. The reduced form of this economy can be described by a two-dimensional differential equation system,

$$\dot{n}(t) = \frac{L}{a} - \frac{\alpha}{v(t)}, \quad \dot{v}(t) = \rho v(t) - \frac{1-\alpha}{n(t)},$$

   where $0 < \alpha < 1$ and $a > 0$. Variables $v(t)$ and $n(t)$ denote the value of the representative firm and the number of firms, respectively. The positive constants ρ and L denote the time preference rate and fix labour supply.

   (a) Draw a phase diagram (for positive $n(t)$ and $v(t)$) and determine the fixpoint.
   (b) What type of fixpoint do you find?

6. Solving linear differential equations
   Solve $\alpha \dot{y}(t) + \beta y(t) = \gamma$, $y(s) = 17$ for

   (a) $t > s$,
   (b) $t < s$.
   (c) What is the forward and what is the backward solution? How do they relate to each other?

7. Comparing forward and backward solutions
   Remember that $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz$ for any well-defined z_1, z_2 and $f(z)$. Replace T by t_0 in (4.3.8) and show that the solution is identical to the one in (4.3.7). Explain why this must be the case.

8. Derivatives of integrals
   Compute the following derivatives.

   (a) $\frac{d}{dy}\int_a^y f(s)\,ds$,

   (b) $\frac{d}{dy}\int_a^y f(s,y)\,ds$,

   (c) $\frac{d}{dy}\int_a^b f(y)\,dy$,

   (d) $\frac{d}{dy}\int f(y)\,dy$.

   (e) Show that the integration by parts formula $\int_a^b \dot{x}y dt = [xy]_a^b - \int_a^b x\dot{y} dt$ holds.

9. Intertemporal and dynamic budget constraints
   Consider the intertemporal budget constraint which equates the discounted expenditure stream to asset holdings plus a discounted income stream,

$$\int_t^\infty D_r(\tau)\,E(\tau)\,d\tau = A(t) + \int_t^\infty D_r(\tau)\,I(\tau)\,d\tau, \tag{4.6.2}$$

   where we defined

$$D_r(\tau) \equiv \exp\left[-\int_t^\tau r(s)\,ds\right]. \tag{4.6.3}$$

   A dynamic budget constraint reads

$$E(t) + \dot{A}(t) = r(t)\,A(t) + I(t). \tag{4.6.4}$$

   (a) Show that solving the dynamic budget constraint yields the intertemporal budget constraint if and only if $\lim_{T\to\infty} A(T)\exp\left[-\int_t^T r(\tau)\,d\tau\right] = 0$.

   (b) Show that differentiating the intertemporal budget constraint yields the dynamic budget constraint.

10. A budget constraint with many assets
    Consider an economy with two assets whose prices are $v_i(t)$. A household owns $n_i(t)$ assets of each type such that total wealth at time t of the household is given by $a(t) = v_1(t)\,n_1(t) + v_2(t)\,n_2(t)$. Each asset pays a flow of dividends $\pi_i(t)$. Let the household earn wage income $w(t)$ and spend $p(t)\,c(t)$ on consumption per unit of time. Show that the household's budget constraint is given by

$$\dot{a}(t) = r(t)\,a(t) + w(t) - p(t)\,c(t)$$

where the interest rates are defined by

$$r(t) \equiv \theta(t) r_1(t) + (1 - \theta(t)) r_2(t), \qquad r_i(t) \equiv \frac{\pi_i(t) + \dot{v}_i(t)}{v_i(t)}$$

and $\theta(t) \equiv v_1(t) n_1(t) / a(t)$ is defined as the share of wealth held in asset 1.

11. Optimal saving
Let optimal saving and consumption behaviour (see ch. 5, e.g. eq. (5.1.6)) be described by the two-dimensional system

$$\dot{c} = gc, \qquad \dot{a} = ra + w - c,$$

where g is the growth rate of consumption, given e.g. by $g = r - \rho$ or $g = (r - \rho)/\sigma$. Solve this system for time paths of consumption c and wealth a.

12. ODE systems
Study transitional dynamics in a two-country world.

(a) Compute time paths for the number $n^i(t)$ of firms in country i. The laws of motion are given by (Grossman and Helpman, 1991; Wälde, 1996)

$$\dot{n}^i = \left(n^A + n^B\right) L^i - n^i \alpha (L + \rho), \quad i = A, B, \quad L = L^A + L^B, \quad \alpha, \rho > 0.$$

Hint: Eigenvalues are $g = (1 - \alpha) L - \alpha \rho > 0$ and $\lambda = -\alpha (L + \rho)$.

(b) Plot the time path of n^A. Choose appropriate initial conditions.

Chapter 5

Finite and infinite horizon models

One widely used approach to solve deterministic intertemporal optimization problems in continuous time consists of using the so-called Hamiltonian function. Given a certain maximization problem, this function can be adapted - just like a recipe - to yield a straightforward result. The first section will provide an introductory example with a finite horizon. It shows how easy it can sometimes be to solve a maximization problem.

It is useful to understand, however, where the Hamiltonian comes from. A list of examples can never be complete, so it helps to be able to derive the appropriate optimality conditions in general. This will be done in the subsequent section. Section 5.4 then discusses what boundary conditions for maximization problems look like and how they can be motivated. The infinite planning horizon problem is then presented and solved in section 5.3 which includes a section on transversality and boundedness conditions. Various examples follow in section 5.5. Section 5.7 finally shows how to work with present-value Hamiltonians and how they relate to current-value Hamiltonians (which are the ones used in all previous sections).

5.1 Intertemporal utility maximization - an introductory example

5.1.1 The setup

Consider an individual that wants to maximize a utility function similar to the one encountered already in (4.4.8),

$$U(t) = \int_t^T e^{-\rho[\tau - t]} \ln c(\tau)\, d\tau. \tag{5.1.1}$$

The planning period starts in t and stops in $T \geq t$. The instantaneous utility function is logarithmic and given by $\ln c(\tau)$. The time preference rate is ρ. The budget constraint of this individual equates changes in wealth, $\dot{a}(\tau)$, to current savings, i.e. the difference between capital and labour income, $r(\tau)a(\tau)+w(\tau)$, and consumption expenditure $c(\tau)$,

$$\dot{a}(\tau) = r(\tau)a(\tau) + w(\tau) - c(\tau). \tag{5.1.2}$$

The maximization task consists of maximizing $U(t)$ subject to this constraint by choosing a path of control variables, here consumption and denoted by $\{c(\tau)\}$.

5.1.2 Solving by optimal control

This maximization problem can be solved by using the present-value or the current-value Hamiltonian. We will work with the current-value Hamiltonian here and in what follows. Section 5.7 presents the present-value Hamiltonian and shows how it differs from the current-value Hamiltonian. The current-value Hamiltonian reads

$$H = \ln c(\tau) + \lambda(\tau)[r(\tau)a(\tau) + w(\tau) - c(\tau)], \tag{5.1.3}$$

where $\lambda(\tau)$ is a multiplier of the constraint. It is called the costate variable as it corresponds to the state variable $a(\tau)$. In maximization problems with more than one state variable, there is one costate variable for each state variable. The costate variable could also be called Hamilton multiplier - similar to the Lagrange multiplier. We show further below that $\lambda(\tau)$ is the shadow price of wealth. The meaning of the terms state, costate and control variables is the same as in discrete time setups.

Omitting time arguments, optimality conditions are

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda = 0, \tag{5.1.4}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial a} = \rho\lambda - r\lambda. \tag{5.1.5}$$

The first-order condition in (5.1.4) is a usual optimality condition: the derivative of the Hamiltonian (5.1.3) with respect to the consumption level c must be zero. The second optimality condition - at this stage - just comes "out of the blue". Its origin will be discussed in a second. Applying logs to the first first-order condition, $-\ln c = \ln \lambda$, and computing derivatives with respect to time yields $-\dot{c}/c = \dot{\lambda}/\lambda$. Inserting into (5.1.5) gives the Euler equation

$$-\frac{\dot{c}}{c} = \rho - r \Leftrightarrow \frac{\dot{c}}{c} = r - \rho. \tag{5.1.6}$$

As this type of consumption problem was first solved by Ramsey in 1928 with some support by Keynes, a consumption rule of this type is often called Keynes-Ramsey rule.

This rule is one of the best-known and most widely used in Economics. It says that consumption increases when the interest rate is higher than the time preference rate. One reason is that a higher interest rate implies - at unchanged consumption levels - a quicker increase in wealth. This is visible directly from the budget constraint (5.1.2). A quicker increase in wealth allows for a quicker increase in consumption. The second reason is that a higher interest rate can lead to a change in the consumption *level* (as opposed to its growth rate). This channel will be analyzed in detail towards the end of ch. 5.6.1.

Equations (5.1.2) and (5.1.6) form a two-dimensional differential equation system in a and c. This system can be solved given two boundary conditions. How these conditions can be found will be treated in ch. 5.4.

5.2 Deriving laws of motion

This subsection shows where the Hamiltonian comes from. More precisely, it shows how the Hamiltonian can be deduced from the optimality conditions resulting from a Lagrange approach. The Hamiltonian can therefore be seen as a shortcut which is quicker than the Lagrange approach but leads to (it needs to lead to) identical results.

5.2.1 The setup

Consider the objective function

$$U(t) = \int_t^T e^{-\rho[\tau - t]} u(y(\tau), z(\tau), \tau)\, d\tau \tag{5.2.1}$$

which we now maximize subject to the constraint

$$\dot{y}(\tau) = Q(y(\tau), z(\tau), \tau). \tag{5.2.2}$$

The function $Q(.)$ is left fairly unspecified. It could be a budget constraint of a household, a resource constraint of an economy or some other constraint. We assume that $Q(.)$ has "nice properties", i.e. it is continuous and differentiable everywhere. The objective function is maximized by an appropriate choice of the path $\{z(\tau)\}$ of control variables.

5.2.2 Solving by the Lagrangian

This problem can be solved by using the Lagrangian

$$\mathcal{L} = \int_t^T e^{-\rho[\tau - t]} u(\tau)\, d\tau + \int_t^T \eta(\tau)\, [Q(\tau) - \dot{y}(\tau)]\, d\tau.$$

The utility function $u(.)$ and the constraint $Q(.)$ are presented as $u(\tau)$ and $Q(\tau)$, respectively. This shortens notation compared to full expressions in (5.2.1) and (5.2.2). The intuition behind this Lagrangian is similar to the one behind the Lagrangian in the discrete time case in (3.7.3) in ch. 3.7, where we also looked at a setup with many constraints. The first part is simply the objective function. The second part refers to the constraints. In the discrete-time case, each point in time had its own constraint with its own Lagrange multiplier. Here, the constraint (5.2.2) holds for a continuum of points τ.

Hence, instead of the sum in the discrete case we now have an integral over the product of multipliers $\eta(\tau)$ and constraints.

This Lagrangian can be rewritten as follows,

$$\begin{aligned}\mathcal{L} &= \int_t^T e^{-\rho[\tau-t]}u(\tau) + \eta(\tau) Q(\tau) d\tau - \int_t^T \eta(\tau)\dot{y}(\tau) d\tau \\ &= \int_t^T e^{-\rho[\tau-t]}u(\tau) + \eta(\tau) Q(\tau) d\tau + \int_t^T \dot{\eta}(\tau) y(\tau) d\tau - [\eta(\tau) y(\tau)]_t^T \end{aligned} \tag{5.2.3}$$

where the last step integrated by parts and $[\eta(\tau) y(\tau)]_t^T$ is the integral function of $\eta(\tau) y(\tau)$ evaluated at T minus its level at t.

Now assume that we could choose not only the control variable $z(\tau)$, but also the state variable $y(\tau)$ at each point in time. The intuition for this is the same as in discrete time in ch. 3.7.2. Hence, we maximize the Lagrangian (5.2.3) with respect to y and z at one particular $\tau \in [t,T]$, i.e. we compute the derivative with respect to $z(\tau)$ and $y(\tau)$.

For the control variable $z(\tau)$, we get a first-order condition

$$e^{-\rho[\tau-t]}u_z(\tau) + \eta(\tau) Q_z(\tau) = 0 \Leftrightarrow u_z(\tau) + e^{\rho[\tau-t]}\eta(\tau) Q_z(\tau) = 0.$$

When we define

$$\lambda(\tau) \equiv e^{\rho[\tau-t]}\eta(\tau), \tag{5.2.4}$$

we find

$$u_z(\tau) + \lambda(\tau) Q_z(\tau) = 0. \tag{5.2.5}$$

For the state variable $y(\tau)$, we obtain

$$\begin{aligned} e^{-\rho[\tau-t]}u_y(\tau) + \eta(\tau) Q_y(\tau) + \dot{\eta}(\tau) = 0 \Leftrightarrow \\ -u_y(\tau) - \eta(\tau) e^{\rho[\tau-t]}Q_y(\tau) = e^{\rho[\tau-t]}\dot{\eta}(\tau). \end{aligned} \tag{5.2.6}$$

Differentiating (5.2.4) with respect to time τ and resinserting (5.2.4) gives

$$\dot{\lambda}(\tau) = \rho e^{\rho[\tau-t]}\eta(\tau) + e^{\rho[\tau-t]}\dot{\eta}(\tau) = \rho\lambda(\tau) + e^{\rho[\tau-t]}\dot{\eta}(\tau).$$

Inserting (5.2.6) and (5.2.4), we obtain

$$\dot{\lambda}(\tau) = \rho\lambda(\tau) - u_y(\tau) - \lambda(\tau) Q_y(\tau). \tag{5.2.7}$$

Equations (5.2.5) and (5.2.7) are the two optimality conditions that solve the above maximization problem jointly with the constraint (5.2.2). We have three equations which fix three variables: The first condition (5.2.5) determines the optimal level of the control variable z. As this optimality condition holds for each point in time τ, it fixes an entire path for z. The second optimality condition (5.2.7) fixes a time path for λ. By letting the costate λ follow an appropriate path, it makes sure, that the level of the state variable (which is not instantaneously adjustable as the maximization of the Lagrangian would suggest) is *as if* it had been optimally chosen at each instant. Finally, the constraint (5.2.2) fixes the time path for the state variable y.

5.2.3 Hamiltonians as a shortcut

Let us now see how Hamiltonians can be justified. The optimal control problem continues to be the one in ch.5.2.1 . Define the Hamiltonian similar to (5.1.3),

$$H = u(\tau) + \lambda(\tau) Q(\tau). \tag{5.2.8}$$

In fact, this Hamiltonian shows the general structure of Hamiltonians. Take the instantaneous utility level (or any other function behind the discount term in the objective function) and add the costate variable λ multiplied by the right-hand side of the constraint. Optimality conditions are then

$$H_z = 0, \tag{5.2.9}$$

$$\dot{\lambda} = \rho\lambda - H_y. \tag{5.2.10}$$

These conditions were already used in (5.1.4) and (5.1.5) in the introductory example in the previous chapter 5.1 and in (5.2.5) and (5.2.7): When the derivatives H_z and H_y in (5.2.9) and (5.2.10) are computed from (5.2.8), this yields equations (5.2.5) and (5.2.7). Hamiltonians are therefore just a shortcut that allow us to obtain results faster than in the case where Lagrangians are used. Note for later purposes that both λ and η have time τ as an argument.

There is an interpretation of the costate variable λ which we simply state at this point (see ch. 6.2 for a formal derivation): The derivative of the objective function with respect to the state variable at t, evaluated on the optimal path, equals the value of the corresponding costate variable at t. Hence, just as in the static Lagrange case, the costate variable measures the change in utility as a result of a change in endowment (i.e. in the state variable). Expressing this formally, define the value function as $V(y(t)) \equiv \max_{\{z(\tau)\}} U(t)$, identical in spirit to the value function in dynamic programming as we got to know it in discrete time. The derivative of the value function, the shadow price $V'(y(t))$, is then the change in utility when behaving optimally resulting from a change in the state $y(t)$. This derivative equals the costate variable, $V'(y(t)) = \lambda$.

5.3 The infinite horizon

5.3.1 Solving by optimal control

- Setup

In the infinite horizon case, the objective function has the same structure as before in e.g. (5.2.1) only that the finite time horizon T is replaced by an infinite time horizon ∞. The constraints are unchanged and the maximization problem reads

$$\max_{\{z(\tau)\}} \int_t^\infty e^{-\rho[\tau-t]} u(y(\tau), z(\tau), \tau) \, d\tau,$$

subject to

$$\dot{y}(\tau) = Q(y(\tau), z(\tau), \tau),$$
$$y(t) = y_t. \tag{5.3.1}$$

We need to assume for this problem that the integral $\int_t^\infty e^{-\rho[\tau-t]}u(.)\,d\tau$ converges for all feasible paths of $y(\tau)$ and $z(\tau)$, otherwise the optimality criterion must be redefined. This boundedness condition is important only in this infinite horizon case. Consider the following figure for the finite horizon case.

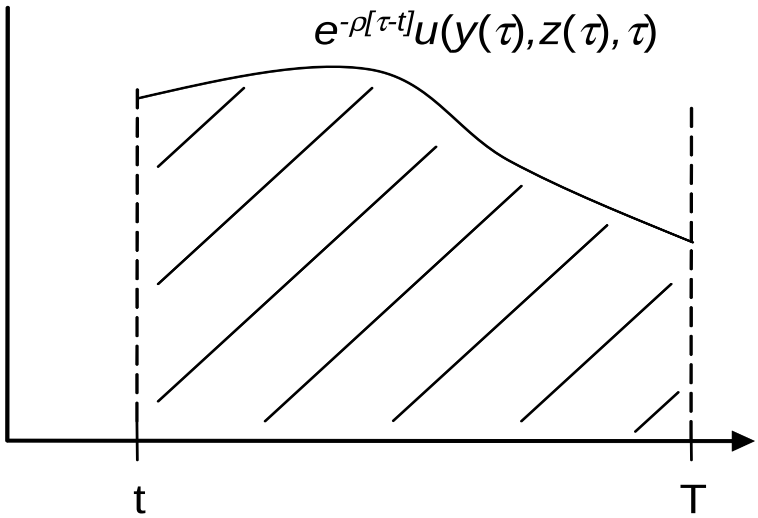


Figure 5.3.1 *Bounded objective function for a finite horizon*

If individuals have a finite horizon (planning starts at t and ends at T) and the utility function $u(.)$ is continuous over the entire planning period (as drawn), the objective function (the shaded area) is finite and the boundedness problem disappears. (As is clear from the figure, the condition of a continuous $u(.)$ could be relaxed.) Clearly, making such an assumption is not always innocuous and one should check, at least after having solved the maximization problem, whether the objective function indeed converges. This will be done in ch. 5.3.2.

- Optimality conditions

The current-value Hamiltonian as in (5.2.8) is defined by $H = u(\tau) + \lambda(\tau) Q(\tau)$. Optimality conditions are (5.2.9) and (5.2.10), i.e.

$$\frac{\partial H}{\partial z} = 0, \qquad \dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial y}.$$

Hence, we have identical optimality conditions to the case of a finite horizon.

5.3.2 The boundedness condition

When introducing an infinite horizon objective function, it was stressed right after (5.3.1) that objective functions must be finite for any feasible paths of the control variable. Otherwise, overall utility $U(t)$ would be infinitely large and there would be no objective for optimizing - we are already infinitely happy! This "problem" of unlimited happiness is particularly severe in models where control variables grow with a constant rate, think e.g. of consumption in a model of growth. A pragmatic approach to checking whether growth is not too high is to first *assume* that it is not too high, then to maximize the utility function and afterwards check whether the initial assumption is satisfied.

As an example, consider the utility function $U(t) = \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau))\, d\tau$. The instantaneous utility function $u(c(\tau))$ is characterized by constant elasticity of substitution as in (2.2.10),

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma}-1}{1-\sigma}, \quad \sigma > 0. \tag{5.3.2}$$

Assume that consumption grows with a rate of g, where this growth rate results from utility maximization. Think of this g as representing e.g. the difference between the interest rate and the time preference rate, corrected by intertemporal elasticity of substitution, as will be found later e.g. in the Keynes-Ramsey rule (5.6.8), i.e. $\dot{c}/c = (r-\rho)/\sigma \equiv g$. Consumption at $\tau \geq t$ is then given by $c(\tau) = c(t)\, e^{g[\tau-t]}$. With this exponential growth of consumption, the utility function becomes

$$U(t) = (1-\sigma)^{-1}\left[c(t)^{1-\sigma}\int_t^\infty e^{-\rho[\tau-t]}e^{(1-\sigma)g[\tau-t]}d\tau + \frac{1}{\rho}\right].$$

This integral is bounded if and only if the boundedness condition

$$(1-\sigma)\, g - \rho < 0$$

holds. This can formally be seen by computing the integral explicitly and checking under which conditions it is finite. Intuitively, this condition makes sense: Instantaneous utility from consumption grows by a rate of $(1-\sigma)\, g$. Impatience implies that future utility is discounted by the rate ρ. Only if this time preference rate ρ is large enough, the overall expression within the integral, $e^{-\rho[\tau-t]}\left(C(\tau)^{1-\sigma}-1\right)$, will fall in τ.

5.4 Boundary conditions and sufficient conditions

So far, maximization problems were presented without boundary conditions. Usually, however, boundary conditions are part of the maximization problem. Without boundary conditions, the resulting differential equation system (e.g. (5.1.2) and (5.1.6) from the introductory example in ch. 5.1) has an infinite number of solutions and the level of control and state variables is not pinned down. We will now consider three cases. All cases will later be illustrated in the phase diagram of section 5.5.

5.4.1 Free value of the state variable at the endpoint

Many problems are of the form (5.2.1) and (5.2.2), where boundary values for the state variable are given by

$$y(t) = y_t, \quad y(T) \text{ free.} \tag{5.4.1}$$

The first condition is the usual initial condition. The second condition allows the state variable to be freely chosen for the end of the planning horizon.

We can use the current-value Hamiltonian (5.2.8) to obtain optimality conditions (5.2.9) and (5.2.10). In addition to the boundary condition $y(t) = y_t$ from (5.4.1), we have (cf. Feichtinger and Hartl, 1986, p. 20),

$$\lambda(T) = 0. \tag{5.4.2}$$

With this additional condition, we have two boundary conditions which allows us to solve our differential equation system (5.2.9) and (5.2.10). This yields a unique solution and an example for this will be discussed further below in ch. 5.5.

5.4.2 Fixed value of the state variable at the endpoint

Now consider (5.2.1) and (5.2.2) with one initial and one terminal condition,

$$y(t) = y_t, \; y(T) = y_T. \tag{5.4.3}$$

In order for this problem to make sense, we assume that a feasible solution exists. This "should" generally be the case, but it is not obvious: Consider again the introductory example in ch. 5.1. Let the endpoint condition be given by "the agent is very rich in T", i.e. $a(T) =$"very large". If $a(T)$ is too large, even zero consumption at each point in time, $c(\tau) = 0 \; \forall \tau \in [t, T]$, would not allow wealth a to be as large as required by $a(T)$. In this case, no feasible solution would exist.

We assume, however, that a feasible solution exists. Optimality conditions are then identical to (5.2.9) and (5.2.10), plus initial and boundary values (5.4.3). Again, two differential equations with two boundary conditions gives level information about the optimal solution and not just information about changes.

The difference between this approach and the previous one is that, now, $y(T) = y_T$ is exogenously given, i.e. part of the maximization problem. Before the corresponding $\lambda(T) = 0$ was endogenously determined as a necessary condition for optimality.

5.4.3 The transversality condition

The analysis of the maximization problem with an infinite horizon in ch. 5.3 also led to a system of two differential equations. One boundary condition is provided by the initial condition in (5.3.1) for the state variable. Hence, again, we need a second condition to pin down the initial level of the control variable, e.g. the initial consumption level.

In the finite horizon case, we had terminal conditions of the type $K(T) = K_T$ or $\lambda(T) = 0$ in (5.4.2) and (5.4.3). As no such terminal T is now available, these conditions need to be replaced by alternative specifications. It is useful to draw a distinction between abstract conditions and conditions which have a practical value in the sense that they can be used to explicitly compute the initial consumption level. A first practically useful condition is the no-Ponzi game condition (4.4.5) resulting from considerations concerning the budget constraint,

$$\lim_{T\to\infty} y(T)\exp\left[-\int_t^T r(\tau)\,d\tau\right] = 0$$

where $r \equiv \partial H/\partial y$. Note that this no-Ponzi game condition can be rewritten as

$$\lim_{T\to\infty} e^{-\rho T}\lambda(T)\,y(T) = \lim_{T\to\infty}\eta(T)\,y(T) = 0, \tag{5.4.4}$$

where the fact that $\dot{\lambda}/\lambda = r - \rho$ implies that $\lambda e^{-\rho t} = \lambda_0 e^{-\int_t^T r(\tau)d\tau}$ was used. The formulations in (5.4.4) of the no-Ponzi game condition is frequently encountered as a second boundary condition. We will use it later in the example of ch. 5.6.1.

A second useful way to determine levels of variables is the existence of a long-run steady state. With a well-defined steady state, one generally analyses properties on the saddle path which leads to this steady state. On this saddle path, the level of variables is determined, at least graphically. Numerical solutions also exist and sometimes analytical closed-form solutions can be found. Often, an analysis of the steady state alone is sufficient. An example where levels of variables are determined when analysing transitional dynamics on the saddle-path leading to the steady state is the central planner problem studied in ch. 5.6.3.

Concerning abstract conditions, a condition occasionally encountered is the transversality condition (TVC),

$$\lim_{t\to\infty}\{\eta(t)\,[y(t) - y^*(t)]\} = 0,$$

where $y^*(t)$ is the path of $y(t)$ for an optimal choice of control variables. There is a considerable literature on the necessity and sufficiency of the TVC and no attempt is made here to cover it. Various references to the literature on the TVC are in section 5.8 on "further reading".

5.4.4 Sufficient conditions

So far, we have only presented conditions that are necessary for a maximum. We do not yet know, however, whether these conditions are also sufficient. Sufficiency can be important, however, as it can be easily recalled when thinking of a static maximization problem. Consider $\max_x f(x)$ where $f'(x^*) = 0$ is necessary for an interior maximum. This is not sufficient as $f'(x^* + \varepsilon)$ could be positive for any $\varepsilon \neq 0$.

For our purposes, necessary conditions are sufficient if either (i) the functions $u(.)$ and $Q(.)$ in (5.2.1) and (5.2.2) are concave in y and z and if $\eta(\tau)$ is positive for all τ, (ii) $Q(.)$ is linear in y and z for any $\eta(t)$ or (iii) $Q(.)$ is convex and $\eta(\tau)$ is negative for all τ.

The concavity of the utility function $u(.)$ and constraint $Q(.)$ can easily be checked and obviously hold, for example, for standard logarithmic and CES utility functions (as in (2.2.10) or (3.9.4)) and for constraints containing a technology as in (5.6.12) below. The sign of the shadow price can be checked by looking at the first order conditions as e.g. (5.1.4) or (5.6.13) later. As we usually assume that utility increases in consumption, we see that - for a problem to make economically sense - the shadow price is positive. Linearity in condition (ii) is often fulfilled when the constraint is e.g. a budget constraint.

See "further reading" on references with a more formal treatment of sufficient conditions.

5.5 Illustrating boundary conditions

Let us now consider an example from microeconomics. We consider a firm that operates under adjustment costs. This will bring us back to phase-diagram analysis, to an understanding of the meaning of fixed and free values of state variables at the end points for a finite planning horizon T, and to the meaning of transversality conditions for the infinite horizon.

5.5.1 A firm with adjustment costs

The maximization problem we are now interested in is a firm that operates under adjustment costs. Capital can not be rented instantaneously on a spot market but its installation is costly. The crucial implication of this simple generalization of the standard theory of production implies that firms "all of a sudden" have an intertemporal and no longer a static optimization problem. As one consequence, factors of production are then no longer paid their value marginal product as in static firm problems.

- The model

A firm maximizes the present value Π_0 of its future instantaneous profits $\pi(t)$,

$$\Pi_0 = \int_0^T e^{-rt}\pi(t)\,dt, \tag{5.5.1}$$

subject to a capital accumulation constraint

$$\dot{K}(t) = I(t) - \delta K(t). \tag{5.5.2}$$

Gross investment $I(t)$ minus depreciation $\delta K(t)$ gives the net increase of the firm's capital stock. Economic reasoning suggests that $K(t)$ should be non-negative. Instantaneous profits are given by the difference between revenue and cost,

$$\pi(t) = pF(K(t)) - \Phi(I(t)). \tag{5.5.3}$$

Revenue is given by $pF(K(t))$ where the production technology $F(.)$ employs capital only. Output increases in capital input, $F'(.) > 0$, but to a decreasing extent, $F''(.) < 0$. The firm's costs are given by the cost function $\Phi(I(t))$. As the firm owns capital, it does not need to pay any rental costs for capital. Costs that are captured by $\Phi(.)$ are adjustment costs which include both the cost of buying and of installing capital. The initial capital stock is given by $K(0) = K_0$. The interest rate r is exogenous to the firm. Maximization takes place by choosing a path of investment $\{I(t)\}$.

The maximization problem is presented slightly differently from previous chapters. We look at the firm from the perspective of a point in time zero and not - as before - from a point in time t. This is equivalent to saying that we normalize t to zero. Both types of objective functions are used in the literature. One with normalization of today to zero as in (5.5.1) and one with a planning horizon starting in t. Economically, this is of no major importance. The presentation with t representing today as normally used in this book is slightly more general and is more useful when dynamic programming is the method chosen for solving the maximization problem. We now use a problem starting in 0 to show that no major differences in the solution techniques arise.

- Solution

This firm obviously has an intertemporal problem, in contrast to the firms we encountered so far. Before solving this problem formally, let us ask where this intertemporal dimension comes from. By looking at the constraint (5.5.2), this becomes clear: Firms can no longer instantaneously rent capital on some spot market. The firm's capital stock is now a state variable and can only change slowly as a function of (positive or negative) investment and depreciation. As an investment decision today has an impact on the capital stock "tomorrow", i.e. the decision today affects future capital levels, there is an intertemporal link between decisions and outcomes at differing points in time.

As discussed after (5.2.8), the current-value Hamiltonian combines the function after the discount term in the objective function, here instantaneous profits $\pi(t)$, with the constraint, here $I(t) - \delta K(t)$, and uses the costate variable $\lambda(t)$. This gives

$$H = \pi(t) + \lambda(t)[I(t) - \delta K(t)] = pF(K(t)) - \Phi(I(t)) + \lambda(t)[I(t) - \delta K(t)].$$

Following (5.2.9) and (5.2.10), optimality conditions are

$$H_I = -\Phi'(I(t)) + \lambda(t) = 0, \tag{5.5.4}$$

$$\begin{aligned}\dot{\lambda}(t) &= r\,\lambda - H_K(t) = r\,\lambda - pF'(K(t)) + \lambda\delta \\ &= (r+\delta)\,\lambda - pF'(K(t)),\end{aligned} \tag{5.5.5}$$

The optimality condition for λ in (5.5.5) shows that the value marginal product of capital, $pF'(K(t))$, still plays a role and it is still compared to the rental price r of capital (the latter being adjusted for the depreciation rate), but there is no longer an equality as in static models of the firm. We will return to this point in exercise 2 of ch. 6.

In a long-run situation where the shadow price is constant, we see that value marginal productivity of capital, $pF'(K(t))$, equals the interest rate plus the depreciation rate corrected for the shadow price. If, by (5.5.4), adjustment costs are given by investment costs only and the price of an investment good is identical to the price of the output good, we are back to equalities known from static models.

Optimality conditions can be presented in a simpler way (i.e. with fewer endogenous variables). First, solve the first optimality condition for the costate variable and compute the time derivative,

$$\dot{\lambda}(t) = \Phi''(I(t))\,\dot{I}(t).$$

Second, insert this into the second optimality condition (5.5.5) to find

$$\Phi''(I(t))\,\dot{I}(t) = (r+\delta)\ \Phi'(I(t)) - pF'(K(t)). \tag{5.5.6}$$

This equation, together with the capital accumulation constraint (5.5.2), is a two-dimensional differential equation system that can be solved, given the initial condition K_0 and one additional boundary condition.

- An example

Now assume adjustment costs are of the form

$$\Phi(I) = v\left[I + I^2/2\right]. \tag{5.5.7}$$

The price to be paid per unit of capital is given by the constant v and costs of installation are given by $I^2/2$. This quadratic term captures the idea that installation costs are low and do not increase quickly, i.e. underproportionally to the new capital stock, at low levels of I but increase overproptionally when I becomes large. Then, optimality requires (5.5.2) and, from inserting (5.5.7) into (5.5.6),

$$\dot{I}(t) = (r+\delta)(1+I(t)) - \frac{p}{v}F'(K(t)) \tag{5.5.8}$$

A phase diagram using (5.5.2) and (5.5.8) is plotted in the following figure. As one can see, we can unambiguously determine that dynamic properties are represented by a saddle-path system with a saddle point as defined in def. 4.2.2.

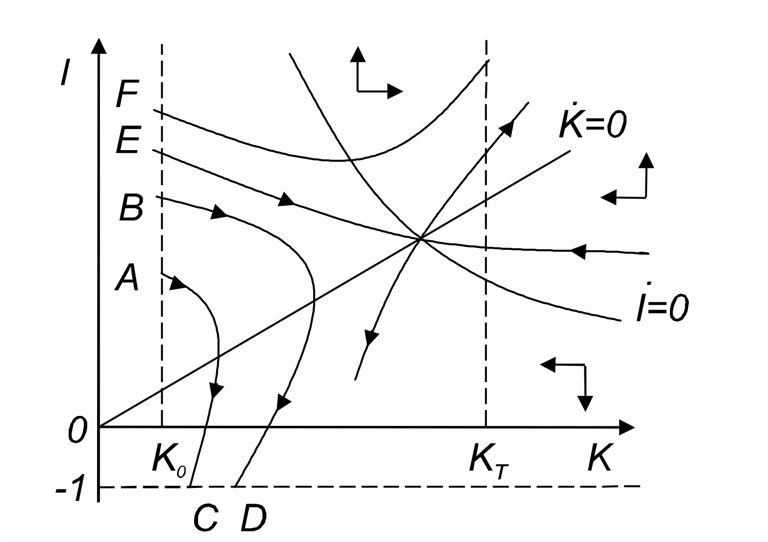


Figure 5.5.1 *A firm with adjustment cost*

We now have to select one of this infinite number of paths that start at K_0. What is the correct investment level I_0? This depends on how we choose the second boundary condition.

5.5.2 Free value at the end point

One modelling opportunity consists of leaving the value of the state variable at the end point open as in ch. 5.4.1. The condition is then $\lambda(T) = 0$ from (5.4.2) which in the context of our example requires

$$\Phi'(I(T)) = 0 \Leftrightarrow v[1 + I(T)] = 0 \Leftrightarrow I(T) = -1.$$

from the first-order condition (5.5.4) and the example for Φ chosen in (5.5.7). In words, the trajectory where the investment level is minus one at T is the optimal one. Economically speaking, this means that the firm will sell capital at T (and also for some time before T) as we will now see.

Let us now see how this information helps us to identify the level of investment and capital, i.e. the corresponding trajectory in the phase diagram by looking at fig. 5.5.1. Look at the trajectory starting at point A first. This trajectory crosses the zero-motion line for capital after some time and eventually hits the horizontal axis where $I = 0$. Have we now found a trajectory which satisfies all optimality conditions? Not yet, as we do not know whether the time needed to go from A to C is exactly of length T. If we started at B and went to D, we would also end up at $I = 0$, also not knowing whether the length is T. Hence, in order to find the appropriate trajectory, a numerical solution is needed. Such a solution would then compute various trajectories as the ones starting at A and B and compare the length required to reach the horizontal axis. When the trajectory requiring T to hit the horizontal axis is found, levels of investment and capital are identified.

There is also hope that searching for the correct trajectory does not take too long. We know that starting on the saddle path which leads to the steady state actually never brings us to the steady state. Capital $K(t)$ and investment $I(t)$ approach the steady state asymptotically but never reach it. As an example, the path for investment would look qualitatively like the graph starting with x_{03} in fig. 4.2.2 which approaches x^* from above but never reaches it. If we start very close to the saddle path, let's say at B, it takes more time to go towards the steady state and then to return than on the trajectory that starts at A. Time to reach the horizontal line is infinity when we start on the saddle path (i.e. we never reach the horizontal line). As time falls, the lower the initial investment level, the correct path can easily be found.

5.5.3 Fixed value at the end point

Let us now consider the case where the end point requires a fixed capital stock K_T. The first aspect to be checked is whether K_T can be reached in the planning period of length T. Maybe K_T is simply too large. Can we see this in our equations?

If we look at the investment equation (5.5.2) only, any K_T can be reached by setting the investment levels $I(T)$ just high enough. When we look at period profits in (5.5.3), however, we see that there is an upper investment level above which profits become negative. If we want to rule out negative profits, investment levels are bounded from above at each point in time by $\pi(t) \geq 0$ and some values at the endpoint K_T are not feasible. The maximization problem would have no solution.

If we now look at a more optimistic example and let K_T not be too high, then we can find the appropriate trajectory in a similar way as before where $I(T) = 0$. Consider the K_T drawn in the figure 5.5.1. When the initial investment level is at point F, the level K_T will be reached faster than on a trajectory that starts between F and E. As time spent between K_0 and K_T is monotonically decreasing, the higher the initial consumption level, i.e. the further the trajectory is away from the steady state, the appropriate initial level can again be easily found by numerical analysis.

5.5.4 Infinite horizon and transversality condition

Let us finally consider the case where the planning horizon is infinity, i.e. we replace T by ∞. Which boundary condition shall we use now? Given the discussion in ch. 5.4.3, one would first ask whether there is some intertemporal constraint. As this is not the case in this model (an example for this will be treated shortly in ch. 5.6.1), one can add an additional requirement to the model analyzed so far. One could require the firm to be in a steady state in the long run. This can be justified by the observation that most firms have a relatively constant size over time. The solution of an economic model of a firm should therefore be characterized by the feature that, in the absence of further shocks, the firm size should remain constant.

The only point where the firm is in a steady state is the intersection point of the zero-motion lines. The question therefore arises where to start at K_0 if one wants to end up in the steady state. The answer is clearly to start on the saddle path. As we are in a deterministic world, a transition towards the steady state would take place and capital and investment approach their long-run values asymptotically. If there were any unanticipated shocks which push the firm off this saddle path, investment would instantaneously be adjusted such that the firm is back on the saddle path.

5.6 Further examples

This section presents further examples of intertemporal optimization problems which can be solved by employing the Hamiltonian. The examples show both how to compute optimality conditions and how to understand the predictions of the optimality conditions.

5.6.1 Infinite horizon - optimal consumption paths

Let us now look at an example with infinite horizon. We focus on the optimal behaviour of a consumer. The problem can be posed in at least two ways. In either case, one part of the problem is the intertemporal utility function

$$U(t) = \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau)) \, d\tau. \tag{5.6.1}$$

Due to the general instantaneous utility function $u(c(\tau))$, it is somewhat more general than e.g. (5.1.1). The second part of the maximization problem is a constraint limiting the total amount of consumption. Without such a constraint, maximizing (5.6.1) would be trivial (or meaningless): With $u'(c(\tau)) > 0$ maximizing the objective simply means setting $c(\tau)$ to infinity. The way this constraint is expressed determines the way in which the problem is solved most straightforwardly.

- Solving by Lagrangian

The constraint to (5.6.1) is given by a budget constraint. The first way in which this budget constraint can be expressed is, again, the intertemporal formulation,

$$\int_t^\infty D_r(\tau) E(\tau) \, d\tau = a(t) + \int_t^\infty D_r(\tau) w(\tau) \, d\tau, \tag{5.6.2}$$

where $E(\tau) = p(\tau) c(\tau)$ and $D_r(\tau) = e^{-\int_t^\tau r(u) du}$. The maximization problem is then given by: maximize (5.6.1) by choosing a path $\{c(\tau)\}$ subject to the budget constraint (5.6.2).

We build the Lagrangean with λ as the time-independent Lagrange multiplier

$$\mathcal{L} = \int_t^\infty e^{-\rho[\tau-t]} u\left(c\left(\tau\right)\right) d\tau \\ - \lambda \left[\int_t^\infty D_r\left(\tau\right) E\left(\tau\right) d\tau - a\left(t\right) - \int_t^\infty D_r\left(\tau\right) w\left(\tau\right) d\tau \right].$$

Note that in contrast to section 5.2.2 where a continuum of constraints implied a continuum of Lagrange multipliers (or, in an alternative interpretation, a time-dependent multiplier), there is only one constraint here.

The optimality conditions are the constraint (5.6.2) and the partial derivative with respect to consumption $c\left(\tau\right)$ at one specific point τ in time, i.e. the first-order condition for the Lagrangian,

$$\mathcal{L}_{c(\tau)} = e^{-\rho[\tau-t]} u'\left(c\left(\tau\right)\right) - \lambda D_r\left(\tau\right) p\left(\tau\right) = 0 \\ \Leftrightarrow D_r\left(\tau\right)^{-1} e^{-\rho[\tau-t]} p\left(\tau\right)^{-1} = \lambda u'\left(c\left(\tau\right)\right)^{-1}. \tag{5.6.3}$$

Note that this first-order condition represents an infinite number of first-order conditions: one for each point τ in time between t and infinity. See ch. 4.3.1 for some background on how to compute a derivative in the presence of integrals. Applying logs to (5.6.3) yields

$$\int_t^\tau r\left(u\right) du - \rho\left[\tau - t\right] - \ln p\left(\tau\right) = \ln \lambda - \ln u'\left(c\left(\tau\right)\right).$$

Differentiating with respect to time τ gives the Keynes-Ramsey rule

$$-\frac{u''\left(c\left(\tau\right)\right)}{u'\left(c\left(\tau\right)\right)} \dot{c}\left(\tau\right) = r\left(\tau\right) - \frac{\dot{p}\left(\tau\right)}{p\left(\tau\right)} - \rho. \tag{5.6.4}$$

The Lagrange multiplier λ drops out as it is not a function of time. Note that $-\frac{u''(c(\tau))}{u'(c(\tau))}$ is Arrow's measure of absolute risk aversion which is a measure of the curvature of the utility function. In our setup of certainty, it is more meaningful, however, to think of the curvature of a utility function as a measure of the intertemporal elasticity of substitution. Even though we are in continuous time now, the intertemporal elasticity of substitution can be defined as in (2.2.9). As we are not in discrete time now, we need to replace the distance of 1 from t to the next period $t+1$ by a period of length Δ. One could then go through the same steps as after (2.2.9) and obtain identical results for a CES and logarithmic instantaneous utility function (see ex. 4).

With a logarithmic utility function, $u\left(c\left(\tau\right)\right) = \ln c\left(\tau\right)$, $u'\left(c\left(\tau\right)\right) = \frac{1}{c(\tau)}$ and $u''\left(c\left(\tau\right)\right) = -\frac{1}{c(\tau)^2}$ and the Keynes-Ramsey rule becomes

$$\frac{\dot{c}\left(\tau\right)}{c\left(\tau\right)} = r\left(\tau\right) - \frac{\dot{p}\left(\tau\right)}{p\left(\tau\right)} - \rho. \tag{5.6.5}$$

- Employing the Hamiltonian

In contrast to above, the utility function (5.6.1) here is maximized subject to the dynamic (or flow) budget constraint,

$$\dot{a}(t) = r(t)\,a(t) + w(t) - p(t)\,c(t). \tag{5.6.6}$$

The solution is obtained by solving exercise 1.

- The interest rate effect on consumption

As an application for the methods we have got to know so far, imagine there is a discovery of a new technology in an economy. All of a sudden, computers or cell-phones or the Internet is available on a large scale. Imagine further that this implies an increase in the returns on investment (i.e. the interest rate r): For any Euro invested, more comes out than before the discovery of the new technology. What is the effect of this discovery on consumption? To ask this more precisely: what is the effect of a change in the interest rate on consumption?

To answer this question, we study a maximization problem as the one just solved, i.e. the objective function is (5.6.1) and the constraint is (5.6.2). We simplify the maximization problem, however, by assuming a CRRA utility function $u(c(\tau)) = \left(c(\tau)^{1-\sigma} - 1\right)/(1-\sigma)$, a constant interest rate and a price being equal to one (imagine the consumption good is the numéraire). This implies that the budget constraint reads

$$\int_t^\infty e^{-r[\tau-t]} c(\tau)\,d\tau = a(t) + \int_t^\infty e^{-r[\tau-t]} w(\tau)\,d\tau \tag{5.6.7}$$

and from inserting the CES utility function into (5.6.4), the Keynes-Ramsey rule becomes

$$\frac{\dot{c}(\tau)}{c(\tau)} = \frac{r-\rho}{\sigma}. \tag{5.6.8}$$

One effect, the growth effect, is straightforward from (5.6.8) or also from (5.6.4). A higher interest rate, ceteris paribus, increases the growth rate of consumption. The second effect, the effect on the *level* of consumption, is less obvious, however. In order to understand it, we undertake the following steps.

First, we solve the linear differential equation in $c(\tau)$ given by (5.6.8). Following ch. 4.3, we find

$$c(\tau) = c(t)\,e^{\frac{r-\rho}{\sigma}(\tau-t)}. \tag{5.6.9}$$

Consumption, starting today in t with a level of $c(t)$, grows exponentially over time at the rate $(r-\rho)/\sigma$ to reach the level $c(\tau)$ at some future $\tau > t$.

In the second step, we insert this solution into the left-hand side of the budget constraint (5.6.7) and find

$$\begin{aligned}\int_t^\infty e^{-r[\tau-t]} c(t)\,e^{\frac{r-\rho}{\sigma}(\tau-t)} d\tau &= c(t)\int_t^\infty e^{-\left(r-\frac{r-\rho}{\sigma}\right)[\tau-t]} d\tau \\ &= c(t)\,\frac{1}{-\left(r-\frac{r-\rho}{\sigma}\right)}\left[e^{-\left(r-\frac{r-\rho}{\sigma}\right)[\tau-t]}\right]_t^\infty.\end{aligned}$$

The simplification stems from the fact that $c(t)$, the initial consumption level, can be pulled out of the term $\int_t^\infty e^{-r[\tau-t]}c(\tau)\,d\tau$, representing the present value of current and future consumption expenditure. Please note that $c(t)$ could be pulled out of the integral also in the case of a non-constant interest rate. Note also that we do not need to know what the level of $c(t)$ is, it is enough to know that there is some $c(t)$ in the solution (5.6.9), whatever its level.

With a constant interest rate, the remaining integral can be solved explicitly. First note that $-\left(r-\frac{r-\rho}{\sigma}\right)$ must be negative. Consumption growth would otherwise exceed the interest rate and a boundedness condition for the objective function similar to the one in ch. 5.3.2 would eventually be violated. (Note, however, that boundedness in ch. 5.3.2 refers to the utility function, here we focus on the present value of consumption.) Hence, we assume $r > \frac{r-\rho}{\sigma} \Leftrightarrow (1-\sigma)r < \rho$. Therefore, for the present value of consumption expenditure we obtain

$$\begin{aligned} c(t)\frac{1}{-\left(r-\frac{r-\rho}{\sigma}\right)}\left[e^{-\left(r-\frac{r-\rho}{\sigma}\right)[\tau-t]}\right]_t^\infty &= c(t)\frac{1}{-\left(r-\frac{r-\rho}{\sigma}\right)}[0-1] \\ &= \frac{c(t)}{r-\frac{r-\rho}{\sigma}} = \frac{\sigma c(t)}{\rho-(1-\sigma)r} \end{aligned}$$

and, inserted into the budget constraint, this yields a closed-form solution for consumption,

$$c(t) = \frac{\rho-(1-\sigma)r}{\sigma}\left\{a(t)+\int_t^\infty e^{-r[\tau-t]}w(\tau)\,d\tau\right\}. \tag{5.6.10}$$

For the special case of a logarithmic utility function, the fraction in front of the curly brackets simplifies to ρ (as $\sigma = 1$).

After these two steps, we have two results, both visible in (5.6.10). One result shows that initial consumption $c(t)$ is a fraction out of wealth of the household. Wealth needs to be understood in a more general sense than usual, however: It is financial wealth $a(t)$ plus, what could be called human wealth (in an economic, i.e. material sense), the present value of labour income, $\int_t^\infty e^{-r[\tau-t]}w(\tau)\,d\tau$. Going beyond t today and realizing that this analysis can be undertaken for any point in time, the relationship (5.6.10) of course holds on any point of an optimal consumption path. The second result is a relationship between the level of consumption and the interest rate, our original question.

We now need to understand the derivative $dc(t)/dr$ in order to further exploit (5.6.10). If we focus only on the term in front of the curly brackets, we find for the change in the level of consumption when the interest rate changes

$$\frac{dc(t)}{dr} = -\frac{1-\sigma}{\sigma}\{.\} \gtreqless 0 \Leftrightarrow \sigma \gtreqless 1.$$

The consumption level increases when the interest rate rises if σ is larger than one, i.e. if the intertemporal elasticity of substitution σ^{-1} is smaller than unity. This is probably the empirically more plausible case (compared to $\sigma < 1$) on the aggregate level. There is micro-evidence, however, where the intertemporal elasticity of substitution can be much larger than unity. This finding is summarized in the following figure.

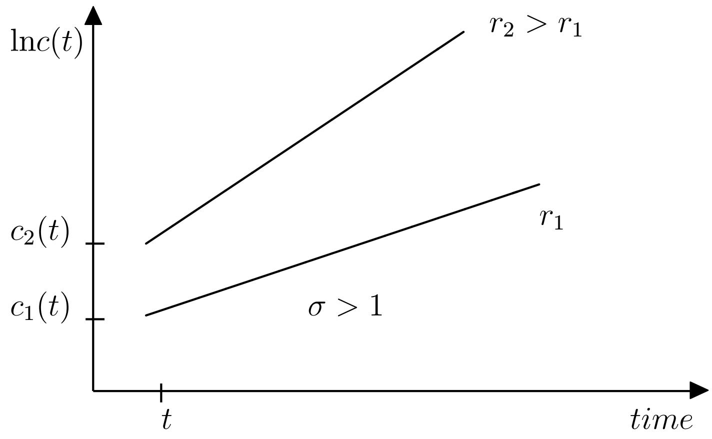


Figure 5.6.1 *The effect of the interest rate on consumption growth and consumption level for an intertemporal elasticity of substitution smaller than one, i.e.* $\sigma > 1$

- The boundary condition for the infinite horizon

The steps we just went through are also an illustration of how to use the no-Ponzi game condition as a condition to obtain level information in an infinite horizon problem. We therefore just saw an example for the discussion in ch. 5.4.3 on transversality conditions.

Solving a maximization problem by Hamiltonian requires a dynamic budget constraint, i.e. a differential equation. The solution is a Keynes-Ramsey rule, also a differential equation. These two differential equations require two boundary conditions in order to obtain a unique solution. One boundary condition is the initial stock of wealth, the second boundary condition is the no-Ponzi game condition.

We just saw how this second condition can indeed be used to obtain level information for the control and the state variable: The No-Ponzi game condition allows us to obtain an intertemporal budget constraint of the type we usually want to work through solving the dynamic budget constraint - see ch. 4.4.2 on "Finding the intertemporal budget constraint". (In the example we just looked at, we did not need to derive an intertemporal budget constraint as it was already given in (5.6.2).) Using the Keynes-Ramsey rule in the way we just did provides the initial consumption level $c(t)$. Hence, by using a boundary condition for this infinite horizon problem, we were able to obtain level information in addition to information on optimal changes.

Note that the principle used here is identical to the one used in the analysis of level effects in ch. 3.4.3 on optimal R&D effort.

5.6.2 Necessary conditions, solutions and state variables

The previous example provides a good opportunity to provide a more in-depth explanation of some concepts that were introduced before.

A distinction was drawn between a solution and a necessary condition in ch. 2.2.2 when discussing (2.2.6). The starting point of our analysis here is the Keynes-Ramsey rule (5.6.8) which is a necessary condition for optimal behaviour. The solution obtained here is given in (5.6.10) and is the outcome of solving the differential equation (5.6.8) and using the intertemporal budget constraint. Looking at these expressions clearly shows that (5.6.8), the outcome of modifying necessary conditions, contains much less information than the solution in (5.6.10). The Keynes-Ramsey rule provides information about the *change* of the control variable "only" while the solution provides information about the *level*.

The solution in (5.6.10) is a closed-form or closed-loop solution. A closed-loop solution is a solution where the control variable is expressed as a function of the state variable and time. In (5.6.10), the state variable is $a(t)$ and the function of time t is the integral $\int_t^\infty e^{-r[\tau-t]}w(\tau)\,d\tau$. Closed-loop solutions stand in contrast to open-loop solutions where the control variable is a function of time only. This distinction becomes meaningful only in a stochastic world. In a deterministic world, any closed-loop solution can be expressed as a function of time only by replacing the state-variable by the function of time which describes its path. When we solve the budget constraint starting at some $a(t_0)$ with $t_0 \leq t$, insert $c(\tau)$ from (5.6.9) into this solution for $a(t)$ and finally insert this solution for $a(t)$ into (5.6.10), we would obtain an expression for the control $c(t)$ as a function of time only.

The solution in (5.6.10) is also very useful for further illustrating the question raised earlier in ch. 3.4.2 on "what is a state variable?". Defining all variables which influence the solution for the control variable as state variable, we clearly see from (5.6.10) that $a(t)$ and the entire path of $w(t)$, i.e. $w(\tau)$ for $t \leq \tau < \infty$ are state variables. As we are in a deterministic world, we can reduce the path of $w(\tau)$ to its initial value in t plus some function of time. What this solution clearly shows is that $a(t)$ is not the only state variable. From solving the maximization problem using the Hamiltonian as suggested after (5.6.6) or from comparing with the similar setup in the introductory example in ch. 5.1, it is sufficient from a practical perspective, however, to take only $a(t)$ as explicit state variable into account. The Keynes-Ramsey rule in (5.1.6) was obtained using the shadow-price of wealth only - see (5.1.5) - but no shadow-price for the wage was required in ch. 5.1.

5.6.3 Optimal growth - the central planner and capital accumulation

- The setup

This example studies the classic central planner problem: First, there is a social welfare function like (5.1.1), expressed slightly more generally as

$$\max_{\{C(\tau)\}} \int_t^\infty e^{-\rho[\tau-t]} u\left(C\left(\tau\right)\right) d\tau.$$

The generalization consists of the infinite planning horizon and the general instantaneous utility function (felicity function) $u\left(c\left(\tau\right)\right)$. We will specify it in the most common version,

$$u\left(C\right) = \frac{C^{1-\sigma} - 1}{1-\sigma}, \tag{5.6.11}$$

where the intertemporal elasticity of substitution is constant and given by $-1/\sigma$.

Second, there is a resource constraint that requires that net capital investment is given by the difference between output $Y\left(K, L\right)$, depreciation δK and consumption C,

$$\dot{K}\left(t\right) = Y\left(K\left(t\right), L\right) - \delta K\left(t\right) - C\left(t\right). \tag{5.6.12}$$

This constraint is valid for t and for all future points in time. Assuming for simplicity that the labour force L is constant, this completely describes this central planner problem.

The planner's choice variable is the consumption level $C\left(\tau\right)$, to be determined for each point in time between today t and the far future ∞. The fundamental trade-off lies in the utility increasing effect of more consumption visible from (5.6.11) and the net-investment decreasing effect of more consumption visible from the resource constraint (5.6.12). As less capital implies less consumption possibilities in the future, the trade-off can also be described as lying in more consumption today vs. more consumption in the future.

- The Keynes-Ramsey rule

Let us solve this problem by employing the Hamiltonian consisting of instantaneous utility plus $\lambda\left(t\right)$ multiplied by the relevant part of the constraint, $H = u\left(C\left(t\right)\right) + \lambda\left(t\right)\left[Y\left(K\left(t\right), L\right) - \delta K\left(t\right) - C\left(t\right)\right]$. Optimality conditions are

$$u'\left(C\left(t\right)\right) = \lambda\left(t\right), \tag{5.6.13}$$

$$\dot{\lambda}\left(t\right) = \rho\lambda\left(t\right) - \frac{\partial H}{\partial K} = \rho\lambda\left(t\right) - \lambda\left(t\right)\left[Y_K\left(K\left(t\right), L\right) - \delta\right].$$

Differentiating the first-order condition (5.6.13) with respect to time gives $u''\left(C\left(t\right)\right)\dot{C}\left(t\right) = \dot{\lambda}\left(t\right)$. Inserting this and (5.6.13) into the second condition again gives, after some rearranging,

$$-\frac{u''\left(C\left(t\right)\right)}{u'\left(C\left(t\right)\right)}\dot{C}\left(t\right) = Y_K\left(K\left(t\right), L\right) - \delta - \rho.$$

This is almost identical to the optimality rule we obtained on the individual level in (5.6.4). The only difference lies in aggregate consumption C instead of c and $Y_K\left(K\left(t\right), L\right) - \delta$ instead of $r\left(\tau\right) - \frac{\dot{p}(\tau)}{p(\tau)}$. Instead of the real interest rate on the household level, we here have the marginal productivity of capital minus depreciation on the aggregate level.

If we assumed a logarithmic instantaneous utility function, $-u''(C(t))/u'(C(t)) = 1/C(t)$ and the Keynes-Ramsey rule would be $\dot{C}(t)/C(t) = Y_K(K(t), L) - \delta - \rho$, similar to (5.1.6) or (5.6.5). In our case of the more general CES specification in (5.6.11), we find $-u''(C(t))/u'(C(t)) = \sigma/C(t)$ such that the Keynes-Ramsey rule reads

$$\frac{\dot{C}(t)}{C(t)} = \frac{Y_K(K(t), L) - \delta - \rho}{\sigma}. \tag{5.6.14}$$

This could be called *the* classic result on optimal consumption in general equilibrium. Consumption grows if marginal productivity of capital exceeds the sum of the depreciation rate and the time preference rate. The higher the intertemporal elasticity of substitution $1/\sigma$, the stronger consumption growth reacts to the differences $Y_K(K(t), L) - \delta - \rho$.

- A phase diagram analysis

The resource constraint of the economy in (5.6.12) plus the Keynes-Ramsey rule in (5.6.14) represent a two-dimensional differential equation system which, given two boundary conditions, give a unique solution for time paths $C(t)$ and $K(t)$. These two equations can be analyzed in a phase diagram. This is probably the phase diagram taught most often in Economics.

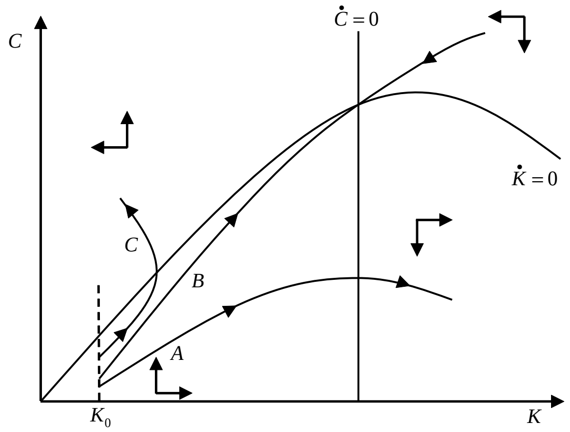


Figure 5.6.2 *Optimal central planner consumption*

Zero-motion lines for capital and labour, respectively, are given by

$$\dot{K}(t) \geq 0 \Leftrightarrow C(t) \leq Y(K(t), L) - \delta K(t), \tag{5.6.15}$$

$$\dot{C}(t) \geq 0 \Leftrightarrow Y_K(K(t), L) \geq \delta + \rho, \tag{5.6.16}$$

when the inequality signs hold as equalities. Zero motion lines are plotted in the above figure.

When consumption lies above the $Y(K(t), L) - \delta K(t)$ line, (5.6.15) tells us that capital decreases, below this line, capital increases. When the marginal productivity of capital is larger than $\delta + \rho$, i.e. when the capital stock is sufficiently small, (5.6.16) tells us that consumption increases. These laws of motion are also plotted in figure 5.6.2. This allows us to draw trajectories A, B and C which all satisfy (5.6.12) and (5.6.14). Hence, again, we have a multitude of solutions for a differential equation system.

As always, boundary conditions allow us to "pick" the single solution to this system. One boundary condition is the initial value K_0 of the capital stock. The capital stock is a state variable and therefore historically given at each point in time. It can not jump. The second boundary condition should fix the initial consumption level C_0. Consumption is a control or choice variable and can therefore jump or adjust to put the economy on the optimal path.

The condition which provides the second boundary condition is, formally speaking, the transversality condition (see ch. 5.4.3). Taking this formal route, one would have to prove that starting with K_0 at the consumption level that puts the economy on path A would violate the TVC or some No-Ponzi game condition. Similarly, it would have to be shown that path C or any path other than B violates the TVC as well. Even though not often admitted, this is not often done in practice. (For an exception to the application of the No-Ponzi game condition, see ch. 5.6.1 or exercise 5 in ch. 5.) Whenever a saddle path is found in a phase diagram, it is argued that the saddle path is the equilibrium path and the initial consumption level is such that the economy finds itself on the path which approaches the steady state. While this is a practical approach, it is also formally satisfied as this path satisfies the TVC indeed.

5.6.4 The matching approach to unemployment

Matching functions are widely used in, for example, Labour economics and Monetary economics. Here we will present the background for their use in labour market modelling.

- Aggregate unemployment

The unemployment rate in an economy is governed by two factors: the speed with which new employment is created and the speed with which existing employment is destroyed. The number of new matches per unit of time dt is given by a matching function. It depends on the number of unemployed U, i.e. the number of those potentially available to fill a vacancy, and the number of vacancies V, $m = m(U, V)$. The number of filled jobs that are destroyed per unit of time is given by the product of the separation rate s and employment, sL. Combining both components, the evolution of the number of unemployed over time is given by

$$\dot{U} = sL - m(U, V). \tag{5.6.17}$$

Defining employment as the difference between the size of the labour force N and the number of unemployed U, $L = N - U$, we obtain

$$\dot{U} = s[N - U] - m(U, V) \Leftrightarrow \dot{u} = s[1 - u] - m(u, V/N),$$

where we divided by the size of the labour force N in the last step to obtain the unemployment rate $u \equiv U/N$. We also assumed constant returns to scale in the matching function $m(.)$. Defining labour market tightness by $\theta \equiv V/U$, the matching function can further be rewritten as

$$m\left(u, \frac{V}{N}\right) = m\left(1, \frac{V}{U}\right)u = \frac{V}{U}m\left(\frac{U}{V}, 1\right)u \equiv \theta q(\theta)u \tag{5.6.18}$$

and we obtain $\dot{u} = s[1-u] - \theta q(\theta)u$ which is equation (1.3) in Pissarides (2000).

Clearly, from (5.6.17), one can easily obtain the evolution of employment, again using the definition $L = N - U$,

$$\dot{L} = m(N-L, V) - sL. \tag{5.6.19}$$

- Optimal behaviour of large firms

Given this matching process, the choice variable of a firm i is no longer employment L_i but the number of vacancies V_i it creates. There is an obvious similarity to the adjustment cost setup in ch. 5.5.1. Here, the firm is no longer able to choose labour directly but only indirectly through vacancies. With adjustment costs, the capital stock is chosen only indirectly through investment.

The firm's objective is to maximize its present value, given by the integral over discounted future profits,

$$\max_{\{V_i, K_i\}} \int_t^\infty e^{-r[\tau-t]}\pi_i(\tau)\,d\tau.$$

Profits are given by

$$\pi_i = Y(K_i, L_i) - rK_i - wL_i - \gamma V_i.$$

Each vacancy implies costs γ measured in units of the output good Y. The firm rents capital K_i from the capital market and pays interest r and a wage rate w per workers. The employment constraint is

$$\dot{L}_i = m(N-L, V)\frac{V_i}{V} - sL_i.$$

The constraint now says - in contrast to (5.6.19) - that only a certain share of all matches goes to the firm under consideration and that this share is given by the share of the firm's vacancies in total vacancies, V_i/V. Alternatively, this says that the "probability" that a match in the economy as a whole fills a vacancy of firm i is given by the number of matches in the economy as a whole divided by the total number of vacancies. As under constant returns to scale for m and using the definition of $q(\theta)$ implicitly in (5.6.18), $m(N-L, V)/V = m(U/V, 1) = q(\theta)$, we can write the firm's constraint as

$$\dot{L}_i = q(\theta)V_i - sL_i. \tag{5.6.20}$$

Assuming small firms, this rate $q(\theta)$ can safely be assumed to be exogenous for the firm's maximization problem.

The current-value Hamiltonian for this problem reads $H = \pi_i + \lambda_i \dot{L}_i$, i.e.

$$H = Y(K_i, L_i) - rK_i - wL_i - \gamma V_i + \lambda_i [q(\theta) V_i - sL_i],$$

and the first-order conditions for capital and vacancies are

$$H_{K_i} = Y_{K_i}(K_i, L_i) - r = 0, \qquad H_{V_i} = -\gamma + \lambda_i q(\theta) = 0. \tag{5.6.21}$$

The optimality condition for the shadow price of labour is

$$\dot{\lambda}_i = r\lambda_i - H_{L_i} = r\lambda_i - (Y_{L_i}(K_i, L_i) - w - \lambda_i s) = (r+s)\lambda_i - Y_{L_i}(K_i, L_i) + w.$$

The first condition for capital is the usual marginal productivity condition applied to capital input. With constant-returns-to-scale production functions, this condition fixes the capital to labour ratio for each firm. This implies that the marginal product of labour is a function of the interest only and therefore identical for all firms, independent of the labour stock, i.e. the size of the firm. This changes the third condition to

$$\dot{\lambda} = (r+s)\,\lambda - Y_L(K, L) + w \tag{5.6.22}$$

which means that the shadow prices are identical for all firms.

The first-order condition for vacancies, written as $\gamma = q(\theta)\,\lambda$ says that the marginal costs γ of a vacancy (which in this special case equal average and unit costs) must be equal to revenue from a vacancy. This expected revenue is given by the share $q(\theta)$ of vacancies that yield a match times the value of a match. The value of a match to the firm is given by λ, the shadow price of labour. The link between λ and the value of an additional unit of labour (or of the state variable, more generally speaking) is analyzed in ch. 6.2.

- General equilibrium

It appears as if the first-order condition for vacancies (5.6.21) was independent of the number of vacancies opened by the firm. In fact, given this structure, the individual firm follows a bang-bang policy. Either the optimal number of vacancies is zero or infinity. In general equilibrium, however, a higher number of vacancies increases the rate $q(\theta)$ in (5.6.20) with which existing vacancies get filled (remember that $\theta = V/U$). Hence, this second condition holds in general equilibrium and we can compute by differentiating with respect to time

$$\dot{\lambda}/\lambda = -\dot{q}(\theta)/q(\theta).$$

The third condition (5.6.22) can then be written as

$$-\frac{\dot{q}(\theta)}{q(\theta)} = r + s - \frac{q(\theta)}{\gamma}[Y_L(K, L) - w] \tag{5.6.23}$$

which is an equation depending on the number of vacancies and employment only. This equation, together with (5.6.19), is a two-dimensional differential equation system which determines V and L, provided there are 2 boundary conditions for V and L and wages and the interest rate are exogenously given. (In a more complete general equilibrium setup, wages would be determined e.g. by Nash-bargaining. For this, we need to derive value functions which is done in ch. 11.2).

- An example for the matching function

Assume the matching function has constant returns to scale and is of the CD type, $m = U^\alpha V^\beta$. As $q(\theta) = m(N-L,V)/V$, it follows that $\frac{\dot{q}(\theta)}{q(\theta)} = \frac{\dot{m}}{m} - \frac{\dot{V}}{V}$. Given our Cobb-Douglas assumption, we can write this as

$$\frac{\dot{q}(\theta)}{q(\theta)} = \alpha\frac{\dot{U}}{U} + \beta\frac{\dot{V}}{V} - \frac{\dot{V}}{V} = \alpha\frac{\dot{U}}{U} - (1-\beta)\frac{\dot{V}}{V}.$$

Hence, the vacancy equation (5.6.23) becomes

$$-\alpha\frac{\dot{U}}{U} + (1-\beta)\frac{\dot{V}}{V} = r + s - \frac{q(\theta)}{\gamma}[Y_L(K,L) - w] \Leftrightarrow$$
$$(1-\beta)\frac{\dot{V}}{V} = r + s - \frac{(N-L)^\alpha}{\gamma V^{1-\beta}}[Y_L(K,L) - w] + \alpha\left[s\frac{L}{N-L} - \frac{V^\beta}{(N-L)^{1-\alpha}}\right]$$

where the last equality used $q(\theta) = (N-L)^\alpha V^{\beta-1}$ and (5.6.17) with $m = U^\alpha V^\beta$. Again, this equation with (5.6.19) is a two-dimensional differential equation system which determines V and L,as just described in the general case.

5.7 The present value Hamiltonian

5.7.1 Problems without (or with implicit) discounting

- The problem and its solution

Let the maximization problem be given by

$$\max_{z(\tau)} \int_t^T F(y(\tau), z(\tau), \tau)\, d\tau \tag{5.7.1}$$

subject to

$$\dot{y}(\tau) = Q(y(\tau), z(\tau), \tau) \tag{5.7.2}$$

$$y(t) = y_t, \quad y(T) \text{ free} \tag{5.7.3}$$

where $y(\tau) \in R^n$, $z(\tau) \in R^m$ and $Q = (Q_1(y(\tau), z(\tau), \tau), Q_2(.), ..., Q_n(.))^T$. A feasible path is a pair $(y(\tau), z(\tau))$ which satisfies (5.7.2) and (5.7.3). $z(\tau)$ is the vector of control variables, $y(\tau)$ is the vector of state variables.

Then define the (present-value) Hamiltonian H^P as

$$H^P = F(\tau) + \eta(\tau) Q(\tau), \tag{5.7.4}$$

where $\eta(\tau)$ is the costate variable,

$$\eta(\tau) = (\eta_1(\tau), \eta_2(\tau), \dots, \eta_n(\tau)) \tag{5.7.5}$$

Necessary conditions for an optimal solution are

$$H_z^P = 0, \tag{5.7.6}$$

$$\dot{\eta}(\tau) = -H_y^P, \tag{5.7.7}$$

and

$$\eta(T) = 0,$$

(Kamien and Schwartz, p. 126) in addition to (5.7.2) and (5.7.3). In order to have a maximum, we need second order conditions $H_{zz} < 0$ to hold (Kamien and Schwartz).

- Understanding its structure

The first m equations (5.7.6) ($z(\tau) \in R^m$) solve the control variables $z(\tau)$ as a function of state and costate variables, $y(\tau)$ and $\eta(\tau)$. Hence

$$z(\tau) = z(y(\tau), \eta(\tau)).$$

The next $2n$ equations (5.7.7) and (5.7.2) ($y(t) \in R^n$) constitute a $2n$ dimensional differential equation system. In order to solve it, one needs, in addition to the n initial conditions given exogenously for the state variables by (5.7.3), n further conditions for the costate variables. These are given by boundary value conditions $\eta(T) = 0$.

Sufficiency of these conditions results from theorem as e.g. in section 5.4.4.

5.7.2 Deriving laws of motion

As in the section on the current-value Hamiltonian, a "derivation" of the Hamiltonian starting from the Lagrange function can be given for the present value Hamiltonian as well.

- The maximization problem

Let the objective function be (5.7.1) that is to be maximized subject to the constraint (5.7.2) and, in addition, a static constraint

$$G(y(\tau), z(\tau), \tau) = 0. \tag{5.7.8}$$

This maximization problem can be solved by using a Lagrangian. The Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \int_t^T F(.) + \eta(\tau)(Q(.) - \dot{y}(\tau)) - \xi(\tau) G(.), t)\, d\tau \\ &= \int_t^T F(.) + \eta(\tau) Q(.) - \xi(\tau) G(.)\, d\tau - \int_\tau^T \eta(\tau)\dot{y}(\tau)\, d\tau\end{aligned}$$

Using the integration by parts rule $\int_a^b \dot{x}ydt = -\int_a^b x\dot{y}dt + [xy]_a^b$ from (4.3.5), integrating the last expression gives

$$\int_\tau^T \eta(\tau)\dot{y}(\tau)\, dt = -\int_t^T \dot{\eta}(\tau) y(\tau)\, d\tau + [\eta(\tau) y(\tau)]_\tau^T$$

and the Lagrangian reads

$$\begin{aligned}\mathcal{L} = \int_t^T & F(.) + \eta(\tau) Q(.) + \dot{\eta}(\tau) y(\tau) - \xi(\tau) G(.)\, d\tau \\ & - [\eta(\tau) y(\tau)]_t^T .\end{aligned}$$

This is now maximized with respect to $y(\tau)$ and $z(\tau)$, both the control and the state variable. We then obtain conditions that are necessary for an optimum.

$$F_z(.) + \eta(\tau) Q_z(.) - \xi(\tau) G_z(.) = 0 \tag{5.7.9}$$

$$F_y(.) + \eta(\tau) Q_y(.) + \dot{\eta}(\tau) - \xi(\tau) G_y(.) = 0 \tag{5.7.10}$$

The last first-order condition can be rearranged to

$$\dot{\eta}(\tau) = -\eta(\tau) Q_y(.) - F_y(.) + \xi(\tau) G_y(.) \tag{5.7.11}$$

These necessary conditions will be the ones used regularly in maximization problems.

- The shortcut

As it is cumbersome to start from a Lagrangian for each dynamic maximization problem, one can define the Hamiltonian as a shortcut as

$$H^P = F(.) + \eta(\tau) Q(.) - \xi(\tau) G(.) . \tag{5.7.12}$$

Optimality conditions are then

$$H_z^P = 0, \tag{5.7.13}$$

$$\dot{\eta}(\tau) = -H_y^P . \tag{5.7.14}$$

which are the same as (5.7.9) and (5.7.11) above and (5.7.6) and (5.7.7) in the last section.

5.7.3 The link between CV and PV

If we solve the problem (5.2.1) and (5.2.2) via the present value Hamiltonian, we would start from

$$H^P(\tau) = e^{-\rho[\tau-t]}G(.) + \eta(\tau) Q(.), \tag{5.7.15}$$

first-order conditions would be

$$\eta(T) = 0, \tag{5.7.16}$$

$$\frac{\partial H^P}{\partial z} = e^{-\rho[\tau-t]}G_z(.) + \eta Q_z(.) = 0, \tag{5.7.17}$$

$$\dot{\eta}(\tau) = -\frac{\partial H^P}{\partial y} = -e^{-\rho[\tau-t]}G_y(.) - \eta Q_y(.) \tag{5.7.18}$$

and we would be done with solving this problem.

- Simplification

We can simplify the presentation of first-order conditions, however, by rewriting (5.7.17) as

$$G_z(\tau) + \lambda(\tau) Q_z(\tau) = 0 \tag{5.7.19}$$

where we used the same definition as in (5.2.4),

$$\lambda(\tau) \equiv e^{\rho[\tau-t]}\eta(\tau). \tag{5.7.20}$$

Note that the argument of the costate variables is always time τ (and not time t).

When we use this definition in (5.7.18), this first-order condition reads

$$e^{\rho[\tau-t]}\dot{\eta}(\tau) = -G_y(.) - \lambda Q_y(.).$$

Replacing the left-hand side by

$$e^{\rho[\tau-t]}\dot{\eta}(\tau) = \dot{\lambda}(\tau) - \rho e^{\rho[\tau-t]}\eta(\tau) = \dot{\lambda}(\tau) - \rho\lambda(\tau).$$

which follows from computing the time τ derivative of the definition (5.7.20), we get

$$\dot{\lambda} = \rho\lambda - G_y - \lambda(\tau)Q_y. \tag{5.7.21}$$

Inserting the definition (5.7.20) into the Hamiltonian (5.7.15) gives

$$e^{\rho[\tau-t]}H^P(t) = G(.) + \lambda Q(.) <\equiv> H^c(t) = G(.) + \lambda Q(.)$$

which defines the link between the present value and the current-value Hamiltonian as

$$H^c(t) = e^{\rho[\tau-t]}H^P(t)$$

- Summary

Hence, instead of first-order conditions (5.7.17) and (5.7.18), we get (5.7.19) and (5.7.21). As just shown these first-order conditions are equivalent.

5.8 Further reading and exercises

The classic reference for the optimal consumption behaviour of an individual household analyzed in ch. 5.6.1 is Ramsey (1928). The paper credits part of the intuitive explanation to Keynes - which led to the name Keynes-Ramsey rule. See Arrow and Kurz (1969) for a detailed analysis.

There are many texts that treat Hamiltonians as a maximization device. Some examples include Dixit (1990, p. 148), Intriligator (1971), Kamien and Schwartz (1991), Leonard and Long (1992) or, in German, Feichtinger and Hartl (1986). Intriligator provides a nice discussion of the distinction between closed- and open-loop controls in his ch. 11.3. The corresponding game-theory definitions for closed-loop and open-loop strategies (which are in perfect analogy) are in Fudenberg and Tirole (1991).

On sufficient conditions, see Kamien and Schwartz (1991, ch. 3 and ch. 15).

The literature on transversality conditions includes Mangasarian (1966), Arrow (1968), Arrow and Kurz (1970), Araujo and Scheinkman (1983), Léonard and Long (1992, p. 288-289), Chiang (1992, p. 217, p. 252) and Kamihigashi (2001). Counterexamples that the TVC is not necessary are provided by Michel (1982) and Shell (1969). See Buiter and Siebert (2007) for a recent very useful discussion and an application.

The issue of boundedness was discovered a relatively long time ago and received renewed attention in the 1990s when the new growth theory was being developed. A more general treatment of this problem was undertaken by von Weizsäcker (1965) who compares utility levels in unbounded circumstances by using "overtaking criteria".

Expressions for explicit solutions for consumption have been known for a while. The case of a logarithmic utility function, i.e. where $\sigma = 1$ and where the fraction in front of the curly brackets in (5.6.10) simplifies to ρ, was obtained by Blanchard (1985).

Vissing-Jørgensen (2002) provides micro-evidence on the level of the intertemporal elasticity of substitution.

Matching models of unemployment go back to Pissarides (1985). For a textbook treatment, see Pissarides (2000).

Exercises chapter 5
Applied Intertemporal Optimization
Hamiltonians

1. Optimal consumption over an infinite horizon
   Solve the maximization problem
   $$\max_{c(\tau)} \int_t^\infty e^{-\rho[\tau-t]} \ln c(\tau)\, d\tau,$$
   subject to
   $$p(t)\, c(t) + \dot{A}(t) = r(t)\, A(t) + w(t).$$
   by
   (a) using the present-value Hamiltonian. Compare the result to (5.6.5).
   (b) Use $u(c(\tau))$ instead of $\ln c(\tau)$.

2. Adjustment costs
   Solve the adjustment cost example for
   $$\Phi(I) = I.$$
   What do optimality conditions mean? What is the optimal end-point value for K and I?

3. Consumption over the life cycle
   The utility of an individual, born at s and living for T periods is given at time t by
   $$u(s,t) = \int_t^{s+T} e^{-\rho[\tau-t]} \ln(c(s,\tau))\, d\tau.$$
   The individual's budget constraint is given by
   $$\int_t^{s+T} D_R(\tau)\, c(s,\tau)\, d\tau = h(s,t) + a(s,t)$$
   where
   $$D_R(\tau) = \exp\left[-\int_t^\tau r(u)\, du\right], \quad h(s,t) = \int_t^{s+T} D_r(\tau)\, w(s,\tau)\, d\tau.$$
   This deplorable individual would like to know how he can lead a happy life but, unfortunately, has not studied optimal control theory!

(a) What would you recommend him? Use a Hamiltonian approach and distinguish between changes of consumption and the initial level. Which information do you need to determine the initial consumption level? What information would you expect this individual to provide you with? In other words, which of the above maximization problems makes sense? Why not the other one?

(b) Assume all prices are constant. Draw the path of consumption in a (t,c(t)) diagram. Draw the path of asset holdings a(t) in the same diagram, by guessing how you would expect it to look. (You could compute it if you want)

4. Optimal consumption

(a) Derive the optimal allocation of expenditure and consumption over time for

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

by employing the Hamiltonian.

(b) Show that this function includes the logarithmic utility function for $\sigma = 1$ (apply L'Hôspital's rule).

(c) Does the utility function $u(c(t))$ make sense for $\sigma > 1$? Why (not)?

(d) Compute the intertemporal elasticity of substitution for this utility function following the discussion after (5.6.4). What is the intertemporal elasticity of substitution for the logarithmic utility function $u(c) = \ln c$?

5. A central planner
You are responsible for the future well-being of hundreds of million of Europeans and centrally plan the EU by assigning a consumption path to each inhabitant. Your problem consists in maximizing a social welfare function of the form

$$U^i(t) = \int_t^\infty e^{-\rho[\tau-t]} \left(C^{1-\sigma} - 1\right)(1-\sigma)^{-1} d\tau$$

subject to the EU resource constraint

$$\dot{K} = BK - C \tag{5.8.1}$$

(a) What are optimality conditions? What is the consumption growth rate?

(b) Under which conditions is the problem well defined (boundedness condition)? Insert the consumption growth rate and show under which conditions the utility function is bounded. Does this condition still allow for positive long-run growth rates?

(c) What is the growth rate of the capital stock? Compute the initial consumption level, by using the no-Ponzi-game condition (5.4.4).

(d) Under which conditions could you resign from your job without making anyone less happy than before?

6. Optimal consumption levels

   (a) Derive a rule for the optimal consumption level for a time-varying interest rate $r(t)$. Show that (5.6.10) can be generalized to

$$c(t) = \frac{1}{\int_t^\infty e^{-\int_t^\tau \frac{\rho-(1-\sigma)r(s)}{\sigma} ds} d\tau} \{W(t) + a(t)\},$$

   where $W(t)$ is human wealth.

   (b) What does this imply for the wealth level $a(t)$?

7. Investment and the interest rate

   (a) Use the result of 5 c) and check under which conditions investment is a decreasing function of the interest rate.

   (b) Perform the same analysis for a budget constraint $\dot{a} = ra + w - c$ instead of (5.8.1).

8. The Ramsey growth model
   Consider a central planer with a standard CES utility function as in ex. 5. Let the resource constraint now be given by

$$\dot{K} = Y(K, L) - C - \delta K$$

   where $Y(.)$ is a neoclassical production function.

   (a) Draw a phase diagram. What is the long-run equilibrium? Perform a stability analysis graphically and analytically (locally).

   (b) Is the utility function bounded?

9. An exam question
   Consider a decentralized economy in continuous time. Factors of production are capital and labour. The initial capital stock is K_0, labour endowment is L. Capital is the only asset, i.e. households can save only by buying capital. Capital can be accumulated also at the aggregate level, $\dot{K}(\tau) = I(\tau) - \delta K(\tau)$. Households have a corresponding budget constraint and a standard intertemporal utility function with time preference rate ρ and infinite planning horizon. Firms produce under perfect competition. Describe such an economy in a formal way and derive its reduced form. Do this step by step:

   (a) Choose a typical production function.

(b) Derive factor demand functions by firms.

(c) Let the budget constraints of households be given by

$$\dot{a}(\tau) = r(\tau)\,a(\tau) + w^L(\tau) - c(\tau).$$

Specify the maximization problem of a household and solve it.

(d) Aggregate the optimal individual decision over all households and describe the evolution of aggregate consumption.

(e) Formulate the goods market equilibrium.

(f) Show that the budget constraint of the household is consistent with the aggregate goods market equilibrium.

(g) Derive the reduced form

$$\dot{K}(\tau) = Y(\tau) - C(\tau) - \delta K(\tau), \quad \frac{\dot{C}(\tau)}{C(\tau)} = \frac{\frac{\partial Y(\tau)}{\partial K(\tau)} - \delta - \rho}{\sigma}$$

by going through these steps and explain the economics behind this reduced form.

Chapter 6

Infinite horizon again

This chapter reanalyzes maximization problems in continuous time that are known from the chapter on Hamiltonians. It shows how to solve them with dynamic programming methods. The sole objective of this chapter is to present the dynamic programming method in a well-known deterministic setup such that its use in a stochastic world in subsequent chapters becomes more accessible.

6.1 Intertemporal utility maximization

We consider a maximization problem that is very similar to the introductory example for the Hamiltonian in section 5.1 or the infinite horizon case in section 5.3. Compared to 5.1, the utility function here is more general and the planning horizon is infinity. None of this is important, however, for understanding the differences in the approach between the Hamiltonian and dynamic programming.

6.1.1 The setup

Utility of the individual is given by

$$U(t) = \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau)) \, d\tau. \tag{6.1.1}$$

Her budget constrained equates wealth accumulation with savings,

$$\dot{a} = ra + w - pc. \tag{6.1.2}$$

The individual can choose the path of consumption $\{c(\tau)\}$ between now and infinity and takes prices and factor rewards as given.

6.1.2 Solving by dynamic programming

As in models of discrete time, the value $V(a(t))$ of the optimal program is defined by the maximum overall utility level that can be reached by choosing the consumption path optimally given the constraint, $V(a(t)) \equiv \max_{\{c(\tau)\}} U(t)$ subject to (6.1.2). When households behave optimally between today and infinity by choosing the optimal consumption path $\{c(\tau)\}$, their overall utility $U(t)$ is given by $V(a(t))$.

- A prelude on the Bellman equation

The derivation of the Bellman equation under continuous time is not as obvious as under discrete time. Even though we do have a today, t, we do not have a clear tomorrow (like a $t+1$ in discrete time). We therefore need to construct a tomorrow by adding a "small" time interval Δt to t. Tomorrow would then be $t+\Delta t$. Note that this derivation is heuristic and more rigorous approaches exist. See e.g. Sennewald (2007) for further references to the literature.

Following Bellman's idea, we rewrite the objective function as the sum of two subperiods,

$$U(t) = \int_t^{t+\Delta t} e^{-\rho[\tau-t]} u(c(\tau))\, d\tau + \int_{t+\Delta t}^{\infty} e^{-\rho[\tau-t]} u(c(\tau))\, d\tau,$$

where Δt is a "small" time interval. As we did in discrete time, we exploit here the additive separability of the objective function. This is the first step of simplifying the maximization problem, as discussed for the discrete-time case after (3.3.4). When we approximate the first integral (think of the area below the function $u(c(\tau))$ plotted over time τ) by $u(c(t))\Delta t$ and the discounting between t and $t+\Delta t$ by $\frac{1}{1+\rho\Delta t}$ and we assume that as of $t+\Delta t$ we behave optimally, we can rewrite the value function $V(a(t)) = \max_{\{c(\tau)\}} U(t)$ as

$$V(a(t)) = \max_{c(t)} \left\{ u(c(t))\Delta t + \frac{1}{1+\rho\Delta t} V(a(t+\Delta t)) \right\}.$$

The assumption of behaving optimally as of $t+\Delta t$ can be seen formally in the fact that $V(a(t+\Delta t))$ now replaces $U(t+\Delta t)$. This is the second step in the procedure to simplify the maximization problem. After these two steps, we are left with only one choice variable $c(t)$ instead of the entire path $\{c(\tau)\}$.When we first multiply this expression by $1+\rho\Delta t$, then divide by Δt and finally move $\frac{V(a(t))}{\Delta t}$ to the right hand side, we get $\rho V(a(t)) = \max_{c(t)} \left\{ u(c(t))[1+\rho\Delta t] + \frac{V(a(t+\Delta t)) - V(a(t))}{\Delta t} \right\}$. Taking the limit $\lim_{\Delta t \to 0}$ gives the Bellman equation,

$$\rho V(a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{dV(a(t))}{dt} \right\}. \tag{6.1.3}$$

This equation again shows Bellman's trick: A maximization problem, consisting of the choice of a path of a choice variable, was broken down to a maximization problem where only the level of the choice variable in t has to be chosen.

The structure of this equation can also be understood from a more intuitive perspective: The term $\rho V(a(t))$ can best be understood when comparing it to rv, capital income at each instant of an individual who owns a capital stock of value v and the interest rate is

r. A household that behaves optimally "owns" the value $V(a(t))$ from optimal behaviour and receives a utility stream of $\rho V(a(t))$. This "utility income" at each instant is given by instantaneous utility from consumption plus the change in the value of optimal behaviour. Note that this structure is identical to the capital-market no-arbitrage condition (4.4.7), $r(t)v(t) = \pi(t) + \dot{v}(t)$ - the capital income stream from holding wealth $v(t)$ on a bank account is identical to dividend payments $\pi(t)$ plus the change $\dot{v}(t)$ in the market price when holding the same level of wealth in stocks.

While the derivation just shown is the standard one, we will now present an alternative approach which illustrates the economic content of the Bellman equation and which is more straightforward. Given the objective function in (6.1.1), we can ask how overall utility $U(t)$ changes over time. To this end, we compute the derivative $dU(t)/dt$ and find (using the Leibniz rule 4.3.3 from ch. 4.3.1)

$$\dot{U}(t) = -e^{-\rho[t-t]}u(c(t)) + \int_t^\infty \frac{d}{dt}e^{-\rho[\tau-t]}u(c(\tau))\,d\tau = -u(c(t)) + \rho U(t).$$

Overall utility $U(t)$ reduces as time goes by by the amount $u(c(t))$ at each instant (as the integral becomes "smaller" when current consumption in t is lost and we start an instant after t) and increases by $\rho U(t)$ (as we gain because future utilities come closer to today when today moves into the future). Rearranging this equation gives $\rho U(t) = u(c(t)) + \dot{U}(t)$. When overall utility is replaced by the value function, we obtain $\rho V(a(t)) = u(c(t)) + \dot{V}(a(t))$ which corresponds in its structure to the Bellman equation (6.1.3).

- DP1: Bellman equation and first-order conditions

We will now follow the three-step procedure to maximization when using the dynamic programming approach as we got to know it in discrete time setups is section 3.3. When we compute the Bellman equation for our case, we obtain for the derivative in (6.1.3) $dV(a(t))/dt = V'(a(t))\dot{a}$ which gives with the budget constraint (6.1.2)

$$\rho V(a(t)) = \max_{c(t)} \{u(c(t)) + V'(a(t))[ra + w - pc]\}. \tag{6.1.4}$$

The first-order condition reads

$$u'(c(t)) = pV'(a(t)) \tag{6.1.5}$$

and makes consumption a function of the state variable, $c(t) = c(a(t))$. In contrast to discrete time models, there is no tomorrow and the interest rate and the time preference rate, present for example in (3.4.5), are absent here. This first-order condition also captures "pros and cons" of more consumption today. The advantage is higher instantaneous utility, the disadvantage is the reduction in wealth. The disadvantage is captured by the change in overall utility due to changes in $a(t)$, i.e. the shadow price $V'(a(t))$, times the price of one unit of consumption in units of the capital good. Loosely speaking, when consumption goes up today by one unit, wealth goes down by p units. Higher consumption increases utility by $u'(c(t))$, p units less wealth reduces overall utility by $pV'(a(t))$.

- DP2: Evolution of the costate variable

In continuous time, the second step of the dynamic programming approach to maximization can be subdivided into two substeps. (i) In the first, we look at the maximized Bellman equation,

$$\rho V(a) = u(c(a)) + V'(a)[ra + w - pc(a)].$$

The first-order condition (6.1.5) together with the maximized Bellman equation determines the evolution of the control variable $c(t)$ and $V(a(t))$. This system can be used as a basis for numerical solution. Again, however, the maximized Bellman equation does not provide very much insight from an analytical perspective. Computing the derivative with respect to $a(t)$ however (and using the envelope theorem) gives an expression for the shadow price of wealth that will be more useful,

$$\begin{aligned} \rho V'(a) &= V''(a)[ra + w - pc] + V'(a) r \Leftrightarrow \\ (\rho - r) V'(a) &= V''(a)[ra + w - pc]. \end{aligned} \tag{6.1.6}$$

(ii) In the second step, we compute the derivative of the costate variable $V'(a)$ with respect to time, giving

$$\frac{dV'(a)}{dt} = V''(a)\dot{a} = (\rho - r) V'(a),$$

where the last equality used (6.1.6). Dividing by $V'(a)$ and using the usual notation $\dot{V}'(a) \equiv dV'(a)/dt$, this can be written as

$$\frac{\dot{V}'(a)}{V'(a)} = \rho - r. \tag{6.1.7}$$

This equation describes the evolution of the costate variable $V'(a)$, the shadow price of wealth.

- DP3: Inserting first-order conditions

The derivative of the first-order condition with respect to time is given by (apply first logs)

$$\frac{u''(c)}{u'(c)}\dot{c} = \frac{\dot{p}}{p} + \frac{\dot{V}'(a)}{V'(a)}.$$

Inserting (6.1.7) gives

$$\frac{u''(c)}{u'(c)}\dot{c} = \frac{\dot{p}}{p} + \rho - r \Leftrightarrow -\frac{u''(c)}{u'(c)}\dot{c} = r - \frac{\dot{p}}{p} - \rho.$$

This is the well-known Keynes Ramsey rule.

6.2 Comparing dynamic programming to Hamiltonians

When we compare optimality conditions under dynamic programming with those obtained when employing the Hamiltonian, we find that they are identical. Observing that our costate evolves when using dynamic programming according to

$$\frac{\dot{V}'(a)}{V'(a)} = \rho - r,$$

as just shown in (6.1.7), we obtain the same equation as we had in the Hamiltonian approach for the evolution of the costate variable, see e.g. (5.1.5) or (5.6.13). Comparing first-order conditions (5.1.4) or (5.2.9) with (6.1.5), we see that they would be identical if we had chosen exactly the same maximization problem. This is not surprising given our applied view of optimization: If there is one optimal path that maximizes some objective function, this one path should always be optimal, independently of which maximization procedure is chosen.

A comparison of optimality conditions is also useful for an alternative purpose, however. As (6.1.7) and e.g. (5.1.5) or (5.6.13) are identical, we can conclude that the derivative $V'(a(t))$ of the value function with respect to the state variable, in our case a, is identical to the costate variable λ in the current-value Hamiltonian approach, $V'(a) = \lambda$. This is where the interpretation for the costate variable in the Hamiltonian approach in ch. 5.2.3 came from. There, we said that the costate variable λ stands for the increase in the value of the optimal program when an additional unit of the state variable becomes available; this is exactly what $V'(a)$ stands for. Hence, the interpretation of a costate variable λ is similar to the interpretation of the Lagrange multiplier in static maximization problems.

6.3 Dynamic programming with two state variables

As a final example for maximization problems in continuous time that can be solved with dynamic programming, we look at a maximization problem with two state variables. Think e.g. of an agent who can save by putting savings on a bank account or by accumulating human capital. Or think of a central planner who can increase total factor productivity or the capital stock. We look here at the first case.

Our agent has a standard objective function,

$$U(t) = \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\, d\tau.$$

It is maximized subject to two constraints. They describe the evolution of the state variables wealth a and human capital h,

$$\dot{a} = f(a, h, c), \qquad \dot{h} = g(a, h, c). \tag{6.3.1}$$

We do not give explicit expressions for the functions $f(.)$ or $g(.)$ but one can think of a standard resource constraint for $f(.)$ as in (6.1.2) and a functional form for $g(.)$ that captures a trade-off between consumption and human capital accumulation: Human capital accumulation is faster when a and h are large but decreases in c. To be precise, we assume that both $f(.)$ and $g(.)$ increase in a and h but decrease in c.

- DP1: Bellman equation and first-order conditions

In this case, the Bellman equation reads

$$\rho V(a,h) = \max\left\{u(c) + \frac{dV(a,h)}{dt}\right\} = \max\left\{u(c) + V_a f(.) + V_h g(.)\right\}.$$

There are simply two partial derivatives of the value function after the $u(c)$ term times the da and dh term, respectively, instead of one as in (6.1.4), where there is only one state variable.

Given that there is still only one control variable, consumption, there is only one first-order condition. This is clearly specific to this example. One could think of a time constraint for human capital accumulation (a trade-off between leisure and learning - think of the Lucas (1988) model) where agents choose the share of their time used for accumulating human capital. In this case, there would be two first-order conditions. Here, however, we have just the one for consumption, given by

$$u'(c) + V_a \frac{\partial f(.)}{\partial c} + V_h \frac{\partial g(.)}{\partial c} = 0 \tag{6.3.2}$$

When we compare this condition with the one-state-variable case in (6.1.5), we see that the first two terms $u'(c) + V_a \frac{\partial f(.)}{\partial c}$ correspond exactly to (6.1.5): If we had specified $f(.)$ as in the budget constraint (6.1.2), the first two terms would be identical to (6.1.5). The third term $V_h \frac{\partial g(.)}{\partial c}$ is new and stems from the second state variable: Consumption now not only affects the accumulation of wealth but also the accumulation of human capital. More consumption gives higher instantaneous utility but, at the same time, decreases future wealth and - now new - the future human capital stock as well.

- DP2: Evolution of the costate variables

As always, we need to understand the evolution of the costate variable(s). In a setup with two state variables, there are two costate variables, or, economically speaking, a shadow price of wealth a and a shadow price of human capital h. This is obtained by partially differentiating the maximized Bellman equation, first with respect to a, then with respect to h. Doing this, we get (employing the envelope theorem right away)

$$\rho V_a = V_{aa} f(.) + V_a \frac{\partial f(.)}{\partial a} + V_{ha} g(.) + V_h \frac{\partial g(.)}{\partial a}, \tag{6.3.3}$$

$$\rho V_h = V_{ah} f(.) + V_a \frac{\partial f(.)}{\partial h} + V_{hh} g(.) + V_h \frac{\partial g(.)}{\partial h}. \tag{6.3.4}$$

As in ch. 6.1, the second step of DP2 consists of computing the time derivatives of the costate variables and in reinserting (6.3.3) and (6.3.4). We first compute time derivatives, inserting (6.3.1) into the last step,

$$\frac{dV_a(a,h)}{dt} = V_{aa}\dot{a} + V_{ah}\dot{h} = V_{aa}f(.) + V_{ah}g(.),$$
$$\frac{dV_h(a,h)}{dt} = V_{ha}\dot{a} + V_{hh}\dot{h} = V_{ha}f(.) + V_{hh}g(.).$$

Then inserting (6.3.3) and (6.3.4), we find

$$\frac{dV_a(a,h)}{dt} = \rho V_a - V_a\frac{\partial f(.)}{\partial a} - V_h\frac{\partial g(.)}{\partial a},$$
$$\frac{dV_h(a,h)}{dt} = \rho V_h - V_a\frac{\partial f(.)}{\partial h} - V_h\frac{\partial g(.)}{\partial h}.$$

The nice feature of this last step is the fact that the cross derivatives V_{ah} and V_{ha} "disappear", i.e. can be substituted out again, using the fact that $f_{xy}(x,y) = f_{yx}(x,y)$ for any twice differentiable function $f(x,y)$. Writing these equations as

$$\frac{\dot{V}_a}{V_a} = \rho - \frac{\partial f(.)}{\partial a} - \frac{V_h}{V_a}\frac{\partial g(.)}{\partial a},$$
$$\frac{\dot{V}_h}{V_h} = \rho - \frac{V_a}{V_h}\frac{\partial f(.)}{\partial h} - \frac{\partial g(.)}{\partial h},$$

allows us to give an interpretation that links them to the standard one-state case in (6.1.7). The costate variable V_a evolves as above, only that instead of the interest rate, we find here $\frac{\partial f(.)}{\partial a} + \frac{V_h}{V_a}\frac{\partial g(.)}{\partial a}$. The first derivative $\partial f(.)/\partial a$ captures the effect a change in a has on the first constraint; in fact, if $f(.)$ represented a budget constraint as in (6.1.2), this would be identical to the interest rate. The second term $\partial g(.)/\partial a$ captures the effect of a change in a on the second constraint; by how much would h increase if there was more a? This effect is multiplied by V_h/V_a, the relative shadow price of h. An analogous interpretation is possible for $\dot{V}_h/V_h$.

- DP3: Inserting first-order conditions

The final step consists of computing the derivative of the first-order condition (6.3.2) with respect to time and replacing the time derivatives $\dot{V}_a$ and $\dot{V}_h$ by the expressions from the previous step DP2. The principle of how to obtain a solution therefore remains unchanged when having two state variables instead of one. Unfortunately, however, it is generally not possible to eliminate the shadow prices from the resulting equation which describes the evolution of the control variable. Some economically suitable assumptions concerning $f(.)$ or $g(.)$ could, however, help.

6.4 Nominal and real interest rates and inflation

We present here a simple model which allows us to understand the difference between the nominal and real interest rate and the determinants of inflation. This chapter is intended (i) to provide another example where dynamic programming can be used and (ii) to show again how to go from a general decentralized description of an economy to its reduced form and thereby obtain economic insights.

6.4.1 Firms, the central bank and the government

- Firms

Firms use a standard neoclassical technology $Y = Y(K, L)$. Producing under perfect competition implies factor demand function of

$$w^K = p\frac{\partial Y}{\partial K}, \quad w^L = p\frac{\partial Y}{\partial L}. \tag{6.4.1}$$

- The central bank and the government

Studying the behaviour of central banks will fill a lot of books. We present the behaviour of the central bank in a very simple way - so simple that what the central bank does here (buy bonds directly from the government) is actually illegal in most OECD countries. Despite this simple presentation, the general result we obtain later would hold in more realistic setups as well.

The central bank issues money $M(t)$ in exchange for government bonds $B(t)$. It receives interest payments iB from the government on the bonds. The balance of the central bank is therefore $iB + \dot{M} = \dot{B}$. This equation says that bond holdings by the central bank increase by $\dot{B}$, either when the central bank issues money $\dot{M}$ or receives interest payments iB on bonds it holds.

The government's budget constraint reads $G + iB = T + \dot{B}$. General government expenditure G plus interest payments iB on government debt B is financed by tax income T and deficit $\dot{B}$. We assume that only the central bank holds government bonds and not private households. Combining the government with the central bank budget therefore yields

$$\dot{M} = G - T.$$

This equation says that an increase in monetary supply is either used to finance government expenditure minus tax income or, if $G = 0$ for simplicity, any monetary increase is given to households in the form of negative taxes, $T = -\dot{M}$.

6.4.2 Households

Household preferences are described by a "money-in-utility" function,

$$\int_t^{\infty} e^{-\rho[\tau - t]} \left[\ln c(\tau) + \gamma \ln \frac{m(\tau)}{p(\tau)} \right] d\tau.$$

In addition to utility from consumption $c(t)$, utility is derived from holding a certain stock $m(t)$ of money, given a price level $p(t)$. Given that wealth of households consists of capital goods plus money, $a(t) = k(t) + m(t)$, and that holding money pays no interest, the household's budget constraint can be shown to read (see exercises)

$$\dot{a} = i[a - m] + w - T/L - pc.$$

Tax payments T/L per representative household are lump-sum. If taxes are negative, T/L represents transfers from the government to households. The interest rate i is defined according to

$$i \equiv \frac{w^K + \dot{v}}{v}.$$

When households choose consumption and the amount of money held optimally, consumption growth follows (see exercises)

$$\frac{\dot{c}}{c} = i - \frac{\dot{p}}{p} - \rho.$$

Money demand is given by (see exercises)

$$m = \gamma \frac{pc}{i}.$$

6.4.3 Equilibrium

- The reduced form

Equilibrium requires equality of supply and demand on the goods market. This is obtained if total supply Y equals demand $C + I$. Letting capital accumulation follow $\dot{K} = I - \delta K$, we get

$$\dot{K} = Y(K, L) - C - \delta K. \tag{6.4.2}$$

This equation determines K. As capital and consumption goods are traded on the same market, this equation implies $v = p$ and the nominal interest rate becomes with (6.4.1)

$$i = \frac{w^K}{p} + \frac{\dot{p}}{p} = \frac{\partial Y}{\partial K} + \frac{\dot{p}}{p}. \tag{6.4.3}$$

The nominal interest rate is given by marginal productivity of capital $w^K/p = \partial Y/\partial K$ (the "real interest rate") plus inflation $\dot{p}/p$.

Aggregating over households yields aggregate consumption growth of $\dot{C}/C = i - \dot{p}/p - \rho$. Inserting (6.4.3) yields

$$\frac{\dot{C}}{C} = \frac{\partial Y}{\partial K} - \rho. \tag{6.4.4}$$

Aggregate money demand is given by

$$M = \gamma \frac{pC}{i}. \tag{6.4.5}$$

Given an exogenous money supply rule and appropriate boundary conditions, these four equations determine the paths of K, C, i and the price level p.

One standard property of models with flexible prices is the dichotomy between real variables and nominal variables. The evolution of consumption and capital - the real side of the economy - is completely independent of monetary influences: Equation (6.4.2) and (6.4.4) determine the paths of K and C just as in the standard optimal growth model without money - see (5.6.12) and (5.6.14) in ch. 5.6.3. Hence, when thinking about equilibrium in this economy, we can think about the real side on the one hand - independently of monetary issues - and about the nominal side on the other hand. Monetary variables have no real effect but real variables have an effect on monetary variables like e.g. inflation.

Needless to say that the real world does not have perfectly flexible prices such that one should expect monetary variables to have an impact on the real economy. This model is therefore a starting point to well understanding structures and not a fully developed model for analysing monetary questions in a very realistic way. Price rigidity would have to be included before doing this.

- A steady state

Assume the technology $Y(K, L)$ is such that in the long-run K is constant. As a consequence, aggregate consumption C is constant as well. Hence, with respect to real variables (including, in addition to K and C, the real interest rate and output), we are in a steady state as in ch. 5.6.3.

Depending on exogenous money supply, equations (6.4.3) and (6.4.5) determine the price level and the nominal interest rate. Substituting the nominal interest rate out, we obtain

$$\frac{\dot{p}}{p} = \gamma \frac{pC}{M} - \frac{\partial Y}{\partial K}.$$

This is a differential equation which is plotted in the next figure. As this figure shows, provided that M is constant, there is a price level p^* which implies that there is zero inflation.

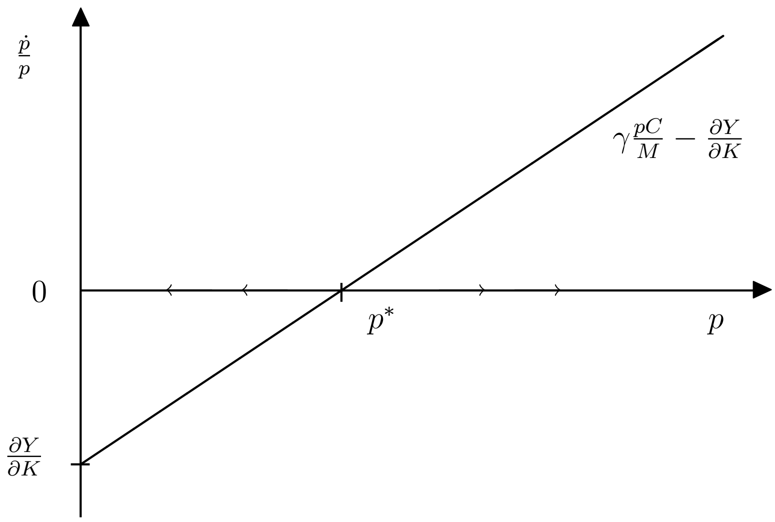


Figure 6.4.1 *The price level p^* in a monetary economy*

Now assume there is money growth, $\dot{M} > 0$. The figure then shows that the price level p^* increases as the slope of the line becomes flatter. Money growth in an economy with constant GDP implies inflation. By looking at (6.4.3) and (6.4.5) again and by focusing on equilibria with constant inflation rates, we know from (6.4.3) that a constant inflation rate implies a constant nominal interest rate. Hence, by differentiating the equilibrium (6.4.5) on the money market, we get

$$\frac{\dot{p}}{p} = \frac{\dot{M}}{M}. \tag{6.4.6}$$

In an economy with constant GDP and increasing money supply, the inflation rate is identical to the growth rate of money supply.

- A growth equilibrium

Now assume there is (exogenous) technological progress at a rate g such that in the long run $\dot{Y}/Y = \dot{C}/C = \dot{K}/K = g$. Then by again assuming a constant inflation rate (implying a constant nominal interest rate) and going through the same steps that led to (6.4.6), we find by differentiating (6.4.5)

$$\frac{\dot{p}}{p} = \frac{\dot{M}}{M} - \frac{\dot{C}}{C}.$$

Inflation is given by the difference between the growth rate of money supply and consumption growth.

- Exogenous nominal interest rates

The current thinking about central bank behaviour differs from the view that the central bank chooses money supply M as assumed so far. The central bank rather sets nominal interest rates and money supply adjusts (where one should keep in mind that money supply is more than just cash used for exchange as modelled here). Can nominal interest rate setting be analyzed in this setup?

Equilibrium is described by equations (6.4.2) to (6.4.5). They were understood to determine the paths of K, C, i and the price level p, given a money supply choice M by the central bank. If one believes that nominal interest rate setting is more realistic, these four equations would simply determine the paths of K, C, M and the price level p, given a nominal interest rate choice i by the central bank. Hence, simply by making an exogenous variable, M, endogenous and making a previously endogenous variable, i, exogenous, the same model can be used to understand the effects of higher and lower nominal interest rates on the economy.

Due to perfect price flexibility, real quantities remain unaffected by the nominal interest rate. Consumption, investment, GDP, the real interest rate, real wages are all determined, as before, by (6.4.2) and (6.4.4) - the dichotomy between real and nominal quantities continues given price flexibility. In (6.4.3), a change in the nominal interest rate affects inflation: high (nominal) interest rates imply high inflation, low nominal interest rates imply low inflation. From the money market equilibrium in (6.4.5) one can then conclude what this implies for money supply, again both for a growing or a stationary economy. Much more needs to be said about these issues before policy implications can be discussed. Any analysis in a general equilibrium framework would however partly be driven by the relationships presented here.

6.5 Further reading and exercises

An alternative way, which is not based on dynamic programming, to reach the same conclusion about the interpretation of the costate variable for Hamiltonian maximization as here in ch. 6.2 is provided by Intriligator (1971, p. 352). An excellent overview and introduction to Monetary economics is provided by Walsh (2003).

Exercises chapter 6
Applied Intertemporal Optimization

Dynamic Programming in Continuous Time

1. The envelope theorem once again
   Compute the derivative (6.1.6) of the maximized Bellman equation without using the envelope theorem.

2. A firm with adjustment costs
   Consider again, as in ch. 5.5.1, a firm with adjustment cost. The firm's objective is

$$\max_{\{I(t),L(t)\}} \int_t^\infty e^{-r[\tau-t]}\pi(\tau)\, d\tau.$$

   In contrast to ch. 5.5.1, the firm now has an infinite planning horizon and employs two factors of production, capital and labour. Instantaneous profits are

$$\pi = pF(K,L) - wL - I - \alpha I^\beta,$$

   where investment I also comprises adjustment costs for $\alpha > 0$. Capital, owned by the firm, accumulates according to $\dot{K} = I - \delta K$. All parameters δ, α, β are constant.

   (a) Solve this maximization problem by using the dynamic programming approach. You may choose appropriate (numerical or other) values for parameters where this simplifies the solution (and does not destroy the spirit of this exercise).
   (b) Show that in the long-run with adjustment costs and at each point in time under the absence of adjustment costs, capital is paid its value marginal product. Why is labour being paid its value marginal product at each point in time?

3. Money in the utility function
   Consider an individual with the following utility function

$$U(t) = \int_t^\infty e^{-\rho[\tau-t]}\left[\ln c(\tau) + \gamma \ln \frac{m(\tau)}{p(\tau)}\right] d\tau.$$

   As always, ρ is the time preference rate and $c(\tau)$ is consumption. This utility function also captures demand for money by including a real monetary stock of

$m(\tau)/p(\tau)$ in the utility function where $m(\tau)$ is the amount of cash and $p(\tau)$ is the price level of the economy. Let the budget constraint of the individual be

$$\dot{a} = i[a - m] + w - T/L - pc.$$

where a is the total wealth consisting of shares in firms plus money, $a = k + m$ and i is the nominal interest rate.

(a) Derive the budget constraint by assuming interest payments of i on shares in firms and zero interest rates on money.

(b) Derive the optimal money demand.

4. Nominal and real interest rates in general equilibrium
Put households from exercise 3 in general equilibrium with capital accumulation and a central bank which chooses money supply M. Compute the real and the nominal interest rate in a long-run equilibrium.

6.6 Looking back

This is the end of part I and II. This is often also the end of a course. This is a good moment to look back at what has been accomplished. After 14 or 15 lectures and the same number of exercise classes, the amount of material covered is fairly impressive.

In terms of maximization tools, this first part has covered

- Solving by substitution
- Lagrange methods in discrete and continuous time
- Dynamic programming in discrete time and continuous time
- Hamiltonian

With respect to model building components, we have learnt

- how to build budget constraints
- how to structure the presentation of a model
- how to derive reduced forms

From an economic perspective, the first part presented

- the two-period OLG model
- the optimal saving central planner model in discrete and continuous time
- the matching approach to unemployment
- the decentralized optimal growth model and
- an optimal growth model with money

Most importantly, however, the tools presented here allow students to "become independent". A very large part of the Economics literature (acknowledging that game theoretic approaches have not been covered here at all) is now open and accessible and the basis for understanding a paper in detail (and not just the overall argument) and for presenting their own arguments in a scientific language are laid out.

Clearly, models with uncertainty present additional challenges. They will be presented and overcome in part III and part IV.

Part III

Stochastic models in discrete time

In part III, the world becomes stochastic. Parts I and II provided many optimization methods for deterministic setups, both in discrete and continuous time. All economic questions that were analyzed were viewed as "sufficiently deterministic". If there was any uncertainty in the setup of the problem, we simply ignored it or argued that it is of no importance for understanding the basic properties and relationships of the economic question. This is a good approach to many economic questions.

Generally speaking, however, real life has few certain components. Death is certain, but when? Taxes are certain, but how high are they? We know that we all exist - but don't ask philosophers. Part III (and part IV later) will take uncertainty in life seriously and incorporate it explicitly in the analysis of economic problems. We follow the same distinction as in part I and II - we first analyse the effects of uncertainty on economic behaviour in discrete time setups in part III and then move to continuous time setups in part IV.

Chapter 7 and 8 are an extended version of chapter 2. As we are in a stochastic world, however, chapter 7 will first spend some time reviewing some basics of random variables, their moments and distributions. Chapter 7 also looks at difference equations. As they are now stochastic, they allow us to understand how distributions change over time and how a distribution converges - in the example we look at - to a limiting distribution. The limiting distribution is the stochastic equivalent to a fix point or steady state in deterministic setups.

Chapter 8 looks at maximization problems in this stochastic framework and focuses on the simplest case of two-period models. A general equilibrium analysis with an overlapping generations setup will allow us to look at the new aspects introduced by uncertainty for an intertemporal consumption and saving problem. We will also see how one can easily understand dynamic behaviour of various variables and derive properties of long-run distributions in general equilibrium by graphical analysis. One can for example easily obtain the range of the long-run distribution for capital, output and consumption. This increases intuitive understanding of the processes at hand tremendously and helps a lot as a guide to numerical analysis. Further examples include borrowing and lending between risk-averse and risk-neutral households, the pricing of assets in a stochastic world and a first look at 'natural volatility', a view of business cycles which stresses the link between jointly endogenously determined short-run fluctuations and long-run growth.

Chapter 9 is then similar to chapter 3 and looks at multi-period, i.e. infinite horizon, problems. As in each chapter, we start with the classic intertemporal utility maximization problem. We then move on to various important applications. The first is a central planner stochastic growth model, the second is capital asset pricing in general equilibrium and how it relates to utility maximization. We continue with endogenous labour supply and the matching model of unemployment. The next section then covers how many maximization problems can be solved without using dynamic programming or the Lagrangian. In fact, many problems can be solved simply by inserting, despite uncertainty. This will be illustrated with many further applications. A final section on finite horizons concludes.

Chapter 7

Stochastic difference equations and moments

Before we look at difference equations in section 7.4, we will first spend a few sections reviewing basic concepts related to uncertain environments. These concepts will be useful at later stages.

7.1 Basics on random variables

Let us first have a look at some basics of random variables. This follows Evans, Hastings and Peacock (2000).

7.1.1 Some concepts

A probabilistic experiment is an occurrence where a complex natural background leads to a chance outcome. The set of possible outcomes of a probabilistic experiment is called the possibility space. A random variable (RV) X is a function which maps from the possibility space into a set of numbers. The set of numbers this RV can take is called the range of this variable X.

The distribution function F associated with the RV X is a function which maps from the range into the probability domain [0,1],

$$F(x) = \text{Prob}(X \leq x).$$

The probability that X has a realization of x or smaller is given by $F(x)$.

We now need to make a distinction between discrete and continuous RVs. When the RV X has a discrete range then $f(x)$ gives finite probabilities and is called the probability function or probability mass function. The probability that X has the realization of x is given by $f(x)$.

When the RV X is continuous, the first derivative of the distribution function F

$$f(x) = \frac{dF(x)}{dx}$$

is called the probability density function f. The probability that the realization of X lies between, say, a and $b > a$ is given by $F(b) - F(a) = \int_a^b f(x)\, dx$. Hence the probability that X equals a is zero.

7.1.2 An illustration

- Discrete random variable

Consider the probabilistic experiment 'tossing a coin twice'. The possibility space is given by $\{HH, HT, TH, TT\}$. Define the RV 'Number of heads'. The range of this variable is given by $\{0, 1, 2\}$. Assuming that the coin falls on either side with the same probability, the probability function of this RV is given by

$$f(x) = \left\{ \begin{array}{c} .25 \\ .5 \\ .25 \end{array} \right\} \quad \text{for} \quad x = \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right. .$$

- Continuous random variable

Think of next weekend. You might consider going to a pub to meet friends. Before you go there, you do not know how much time you will spend there. If you meet a lot of friends, you will stay longer; if you drink just one beer, you will leave soon. Hence, going to a pub on a weekend is a probabilistic experiment with a chance outcome.

The set of possible outcomes with respect to the amount of time spent in a pub is the possibility space. Our random variable T maps from this possibility space into a set of numbers with a range from 0 to, let's say, 4 hours (as the pub closes at 1 am and you never go there before 9 p.m.). As time is continuous, $T \in [0, 4]$ is a continuous random variable. The distribution function $F(t)$ gives you the probability that you spend a period of length t or shorter in the pub. The probability that you spend between 1.5 and two hours in the pub is given by $\int_{1.5}^{2} f(t)\, dt$, where $f(t)$ is the density function $f(t) = dF(t)/dt$.

7.2 Examples for random variables

We now look at some examples of RVs that are useful for later applications. As an RV is completely characterized by its range and its probability or density function, we will describe RVs by providing this information. Many more random variables exist than those presented here and the interested reader is referred to the "further reading" section at the end.

7.2.1 Discrete random variables

- Discrete uniform distribution

range	$x \in \{a, a+1, ..., b-1, b\}$
probability function	$f(x) = 1/(b-a+1)$

An example for this RV is the die. Its range is 1 to 6, the probability for any number (at least for fair dice) is 1/6.

- The Poisson distribution

range	$x \in \{0,\ 1,\ 2, ...\}$
probability function	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Here λ is some positive parameter. When we talk about stochastic processes in part IV, this will be called the arrival rate. An example for this RV is e.g. the number of falling stars visible on a warm summer night at a nice beach.

7.2.2 Continuous random variables

- Normal distribution

range	$x \in]-\infty, +\infty[$
density function	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

The mean and the standard deviation of X are given by μ and σ.

- Standard normal distribution

This is the normal distribution with mean and standard deviation given by $\mu = 0$ and $\sigma = 1$.

- Exponential distribution

range	$x \in [0, \infty[$
density function	$f(x) = \lambda e^{-\lambda x}$

Again, λ is some positive parameter. One standard example for x is the duration of unemployment for an individual who just lost his or her job.

7.2.3 Higher-dimensional random variables

So far we have studied one-dimensional RVs. In what follows, we will occasionally work with multi-dimensional RVs as well. For our illustration purposes here it will suffice to focus on a two-dimensional normal distribution. Consider two random variables X_1 and X_2. They are (jointly) normally distributed if the density function of X_1 and X_2 is given by

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\tilde{x}_1^2 - 2\rho\tilde{x}_1\tilde{x}_2 + \tilde{x}_2^2\right)}, \tag{7.2.1}$$

$$\text{where } \tilde{x}_i = \frac{x_i - \mu_i}{\sigma_i}, \quad \rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}. \tag{7.2.2}$$

The mean and standard deviation of the RVs are denoted by μ_i and σ_i. The parameter ρ is called the correlation coefficient between X_1 and X_2 and is defined as the covariance σ_{12} divided by the standard deviations (see ch. 7.3.1).

The nice aspects about this two-dimensional normally distributed RV (the same holds for n-dimensional RVs) is that the density function of each individual RV X_i, i.e. the marginal density, is given by the standard expression which is independent of the correlation coefficient,

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2}\tilde{x}_i^2}. \tag{7.2.3}$$

This implies a very convenient way to go from independent to correlated RVs in a multi-dimensional setting: When we want to assume independent normally distributed RVs, we assume that (7.2.3) holds for each random variable X_i and set the correlation coefficient to zero. When we want to work with dependent RVs that are individually normally distributed, (7.2.3) holds for each RV individually as well but, in addition, we fix a non-zero coefficient of correlation ρ.

7.3 Expected values, variances, covariances and all that

Here we provide various definitions, some properties and results on transformations of RVs. Only in some selected cases do we provide proofs. A more in depth introduction can be found in many textbooks on statistics.

7.3.1 Definitions

For definitions, we shall focus on continuous random variables. For discrete random variables, the integral is replaced by a sum - in very loose notation, $\int_a^b g(x)\,dx$ is replaced by $\Sigma_{i=a}^b g(x_i)$, where a and b are constants which can be minus or plus infinity and $g(x)$ is some function. In the following definitions, $\int .dx$ means the integral over the relevant range (i.e. from minus to plus infinity or from the lower to upper bound of the range of the RV under consideration).

Mean	$EX \equiv \int x f(x)\, dx$
	$z = g(x,\, y) \Rightarrow EZ = \int\int g(x,\, y)\, f(x,\, y)\, dx\, dy$
Variance	$\text{var}X \equiv \int (x - EX)^2 f(x)\, dx$
kth uncentered moment	$EX^k \equiv \int x^k f(x)\, dx$
kth centered moment	$E(X - EX)^k \equiv \int (x - EX)^k f(x)\, dx$
covariance	$\text{cov}(X, Y) \equiv \int\int (x - EX)(y - EY)\, f(x,\, y)\, dx\, dy$
correlation coefficient	$\rho_{XY} \equiv \text{cov}(X, Y) / \sqrt{\text{var}X\ \text{var}Y}$
independence	$p(X \in A,\, Y \in B) = P(X \in A)\, P(Y \in B)$

Table 7.3.1 *Some basic definitions*

7.3.2 Some properties of random variables

- Basic

Here are some useful properties of random variables. They are listed here for later reference. More background can be found in many statistics textbooks.

$$
\begin{aligned}
&E[a + bX] = a + bEX \\
&E[bX + cY] = bEX + cEY \\
&E(XY) = EXEY + \text{cov}(X,\, Y)
\end{aligned}
$$

Table 7.3.2 *Some properties of expectations*

$$
\begin{aligned}
\text{var}X &= E\left[(X - EX)^2\right] = E\left[X^2 - 2XEX + (EX)^2\right] \\
&= EX^2 - 2(EX)^2 + (EX)^2 = EX^2 - (EX)^2
\end{aligned} \tag{7.3.1}
$$

$$
\begin{aligned}
\text{var}(a + bX) &= b^2 \text{var}X \\
\text{var}(X + Y) &= \text{var}X + \text{var}Y + 2\text{cov}(X,\, Y)
\end{aligned} \tag{7.3.2}
$$

Table 7.3.3 *Some properties of variances*

$$
\begin{aligned}
\text{cov}(X, X) &= \text{var}X \\
\text{cov}(X, Y) &= E(XY) - EXEY \\
\text{cov}(a + bX, c + dY) &= bd\, \text{cov}(X,\, Y)
\end{aligned} \tag{7.3.3}
$$

Table 7.3.4 *Some properties of covariances*

- Advanced

Here we present a theorem which is very intuitive and highly useful for analytically studying the pro- and countercyclical behaviour of endogenous variables in models of business cycles. The theorem says that if two variables depend "in the same sense" on some RV (i.e. they both increase or decrease in the RV), then these two variables have a positive covariance. If e.g. both GDP and R&D expenditure increase in TFP and TFP is random, then GDP and R&D expenditure are procyclical.

Theorem 7.3.1 *Let X be a random variable and $f(X)$ and $g(X)$ two functions such that $f'(X)g'(X) \gtreqless 0 \; \forall \; x \in X$. Then*

$$cov(f(X), g(X)) \gtreqless 0.$$

Proof. We only prove the "> 0" part. We know from (7.3.3) that $\text{cov}(Y, Z) = E(YZ) - EYEZ$. With $Y = f(X)$ and $Z = g(X)$, we have

$$\begin{aligned} \text{cov}(f(X), g(X)) &= E(f(X)g(X)) - Ef(X)Eg(X) \\ &= \int f(x)g(x)p(x)\,dx - \int f(x)p(x)\,dx \int g(x)p(x)\,dx, \end{aligned}$$

where $p(x)$ is the density of X. Hence

$$\text{cov}(f(X), g(X)) > 0 \Leftrightarrow \int f(x)g(x)p(X)\,dx > \int f(x)p(x)\,dx \int g(x)p(x)\,dx.$$

The last inequality holds for $f'(X)g'(X) > 0$ as shown by Čebyšev and presented in Mitrinović (1970, Theorem 10, sect. 2.5, p. 40). ■

7.3.3 Functions on random variables

We will occasionally encounter the situation where we need to compute density functions of functions of RVs. Here are some examples.

- Linearly transforming a normally distributed RV

Consider a normally distributed RV $X \sim N(\mu, \sigma^2)$. What is the distribution of the $Y = a + bX$? We know from ch. 7.3.2 that for any RV, $E(a + bX) = a + bEX$ and $Var(a + bX) = b^2 Var(X)$. As it can be shown that a linear transformation of a normally distributed RV gives a normally distributed RV again, Y is also normally distributed with $Y \sim N(a + b\mu, b^2\sigma^2)$.

- An exponential transformation

Consider the RV $Y = e^X$ where X is again normally distributed. The RV Y is then lognormally distributed. A variable is lognormally distributed if its logarithm is normally distributed, $\ln Y \sim N(\mu, \sigma^2)$. The mean and variance of this distribution are given by

$$\mu_y = e^{\mu + \frac{1}{2}\sigma^2}, \quad \sigma_y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right). \tag{7.3.4}$$

Clearly, Y can only have non-negative realizations.

- Transformations of lognormal distributions

Let there be two lognormally distributed variables Y and Z. Any transformation of the type Y^α, where α is a constant, or products like YZ are also lognormally distributed.

To show this, remember that we can express Y and Z as $Y = e^{X_1}$ and $Z = e^{X_2}$, with the X_i being (jointly) normally distributed. Hence, for the first example, we can write $Y^\alpha = e^{\alpha X_1}$. As αX_1 is normally distributed, Y^α is lognormally distributed. For the second, we write $YZ = e^{X_1}e^{X_2} = e^{X_1+X_2}$. As the X_i are (jointly) normally distributed, their sum is as well and YZ is lognormally distributed.

Total factor productivity is sometimes argued to be lognormally distributed. Its logarithm is then normally distributed. See e.g. ch. 8.1.6.

- The general case

Consider now a general transformation of the type $y = y(x)$ where the RV X has a density $f(x)$. What is the density of Y? The answer comes from the following

Theorem 7.3.2 *Let X be a random variable with density $f(x)$ and range $[a, b]$ which can be $]-\infty, +\infty[$. Let Y be defined by the monotonically increasing function $y = y(x)$. Then the density $g(y)$ is given by $g(y) = f(x(y))\frac{dx}{dy}$ on the range $[y(a), y(b)]$.*

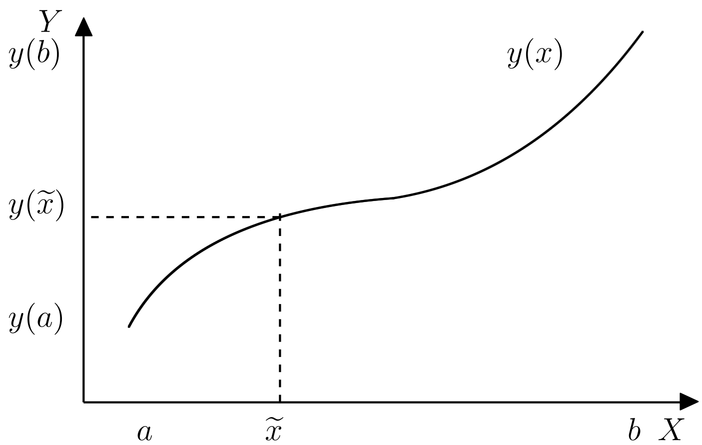


Figure 7.3.1 *Transforming a random variable*

This theorem can be easily proven as follows. The proof is illustrated in fig. 7.3.1. The figure plots the RV X on the horizontal and the RV Y on the vertical axis. A monotonically increasing function $y(x)$ represents the transformation of realizations x into y.

Proof. The transformation of the range is immediately clear from the figure: When X is bounded between a and b, Y must be bounded between $y(a)$ and $y(b)$. The proof for the density of Y requires a few more steps: The probability that y is smaller than some $y(\tilde{x})$ is identical to the probability that X is smaller than this $\tilde{x}$. This follows from the monotonicity of the function $y(x)$. As a consequence, the distribution function (cumulative density function) of Y is given by $G(y) = F(x)$ where $y = y(x)$ or, equivalently, $x = x(y)$. The derivative then gives the density function, $g(y) \equiv \frac{d}{dy}G(y) = \frac{d}{dy}F(x(y)) = f(x(y))\frac{dx}{dy}$. ■

7.4 Examples of stochastic difference equations

We now return to the first main objective of this chapter, the description of stochastic processes through stochastic difference equations.

7.4.1 A first example

- The difference equation

Possibly the simplest stochastic difference equation is the following

$$x_t = ax_{t-1} + \varepsilon_t, \tag{7.4.1}$$

where a is a positive constant and the stochastic component ε_t is distributed according to some distribution function over a range which implies a mean μ and a variance σ^2, $\varepsilon_t \sim (\mu, \sigma^2)$. We do not make any specific assumption about ε_t at this point. Note that the stochastic components ε_t are i.i.d. (identically and independently distributed) which implies that the covariance between any two distinct ε_t is zero, $\text{cov}(\varepsilon_t, \varepsilon_s) = 0$ $\forall t \neq s$. An alternative representation of x_t with identical distributional properties would be $x_t = ax_{t-1} + \mu + v_t$ with $v_t \sim (0, \sigma^2)$.

- Solving by substitution

In complete analogy to deterministic difference equations in ch. 2.5.3, equation (7.4.1) can be solved for x_t as a function of time and past realizations of ε_t, provided we have a boundary condition x_0 for $t = 0$. By repeated reinserting, we obtain

$$x_1 = ax_0 + \varepsilon_1, \quad x_2 = a[ax_0 + \varepsilon_1] + \varepsilon_2 = a^2x_0 + a\varepsilon_1 + \varepsilon_2,$$
$$x_3 = a\left[a^2x_0 + a\varepsilon_1 + \varepsilon_2\right] + \varepsilon_3 = a^3x_0 + a^2\varepsilon_1 + a\varepsilon_2 + \varepsilon_3$$

and eventually

$$\begin{aligned} x_t &= a^tx_0 + a^{t-1}\varepsilon_1 + a^{t-2}\varepsilon_2 + ... + \varepsilon_t \\ &= a^tx_0 + \Sigma_{s=1}^t a^{t-s}\varepsilon_s. \end{aligned} \tag{7.4.2}$$

- The mean

A stochastic difference equation does not predict how the stochastic variable x_t for $t > 0$ actually evolves, it only predicts the distribution of x_t for all future points in time. The realization of x_t is random. Hence, we can only hope to understand something about the distribution of x_t. To do so, we start by analyzing the mean of x_t for future points in time.

Denote the conditional expected value of x_t by $\tilde{x}_t$,

$$\tilde{x}_t \equiv E_0 x_t, \tag{7.4.3}$$

i.e. $\tilde{x}_t$ is the value expected when we are in $t = 0$ for x_t in $t > 0$. The expectations operator E_0 is conditional, i.e. it uses our "knowledge" in 0 when we compute the expectation for x_t. The "0" says that we have all information for $t = 0$ and know therefore x_0 and ε_0, but we do not know ε_1, ε_2, etc. Put differently, we look at the conditional distribution of x_t: What is the mean of x_t conditional on x_0?

By applying the expectations operator E_0 to (7.4.1), we obtain

$$\tilde{x}_t = a\tilde{x}_{t-1} + \mu.$$

This is a deterministic difference equation which describes the evolution of the expected value of x_t over time. There is again a standard solution to this equation which reads

$$\tilde{x}_t = a^t x_0 + \mu \Sigma_{s=1}^t a^{t-s} = a^t x_0 + \mu \frac{1 - a^t}{1 - a}, \tag{7.4.4}$$

where we used $\Sigma_{s=1}^t a^{t-s} = a^0 + a^1 + ... + a^{t-1} = \Sigma_{i=0}^{t-1} a^i$ and, from ch. 2.5.1, $\Sigma_{i=0}^n a^i = \left(1 - a^{n+1}\right) / \left(1 - a\right)$. This equation shows that the expected value of x_t changes over time as t changes. Note that $\tilde{x}_t$ might increase or decrease (see the exercises).

- The variance

Let us now look at the variance of x_t. We obtain an expression for the variance by starting from (7.4.2) and observing that the terms in (7.4.2) are all independent from each other: $a^t x_0$ is a constant and the disturbances ε_s are i.i.d. by assumption. The variance of x_t is therefore given by the sum of variances (compare (7.3.2) for the general case including the covariance),

$$Var\left(x_t\right) = 0 + \Sigma_{s=1}^t \left(a^{t-s}\right)^2 Var\left(\varepsilon_s\right) = \sigma^2 \Sigma_{s=1}^t \left(a^2\right)^{t-s} = \sigma^2 \Sigma_{i=0}^{t-1} \left(a^2\right)^i = \sigma^2 \frac{1 - a^{2t}}{1 - a^2}. \tag{7.4.5}$$

We see that it is also a function of t. The fact that, for $0 < a < 1$, the variance becomes larger the higher t appears intuitively clear. The further we look into the future, the "more randomness" there is: equation (7.4.2) shows that a higher t means that more random variables are added up. (One should keep in mind, however, that i.i.d. variables are added up. If they were negatively correlated, the variance would not necessarily increase over time.)

The fact that we used $Var\left(a^t x_0\right) = 0$ here also shows that we work with a conditional distribution of x_t. We base our computation of the variance at some future point in time on our knowledge of the RV X in $t = 0$. If we wanted to make this point very clearly, we could write $Var_0\left(x_t\right)$.

- Long-run behaviour

When we considered deterministic difference equations like (2.5.6), we found the long-run behaviour of x_t by computing the solution of this difference equation and by letting t approach infinity. This would give us the fixpoint of this difference equation as e.g. in (2.5.7). The concept that corresponds to a fixpoint/ steady state in an uncertain environment, e.g. when looking at stochastic difference equations like (7.4.1), is the limiting distribution. As stated earlier, stochastic difference equations do not tell us anything about the evolution of x_t itself, they only tell us something about the *distribution* of x_t. It would therefore make no sense to ask what x_t is in the long run - it will always remain random. It makes a lot of sense, however, to ask what the distribution of x_t is for the long run, i.e. for $t \to \infty$.

All results so far were obtained without a specific distributional assumption for ε_t apart from specifying a mean and a variance. Understanding the long-run distribution of x_t in (7.4.1) is easy if we assume that the stochastic component ε_t is normally distributed for each t, $\varepsilon_t \sim N\left(\mu, \sigma^2\right)$. In this case, starting in $t = 0$, the variable x_1 is from (7.4.1) normally distributed as well. As a weighted sum of two random variables that are (jointly) normally distributed gives again a random variable with a normal distribution (where the mean and variance can be computed as in ch. 7.3.2), we also know from (7.4.1) that x_1, x_2 ..., x_t, ... are all normally distributed.

As the normal distribution is a two-parameter distribution characterized by the mean and variance, we can find a stable distribution of x_t for the long-run if the mean and variance approach some constant. Neglecting the cases of $a \geq 1$, the mean $\tilde{x}$ of our long-run normal distribution is given from (7.4.4) by the fixpoint

$$\lim_{t\to\infty} \tilde{x}_t \equiv \tilde{x}_\infty = \mu \frac{1}{1-a}, \quad 0 < a < 1.$$

The variance of the long-run distribution is from (7.4.5)

$$\lim_{t\to\infty} var\left(x_t\right) = \sigma^2 \frac{1}{1-a^2}.$$

Hence, the long-run distribution of x_t for $0 < a < 1$ is a normal distribution with mean $\mu/\left(1-a\right)$ and variance $\sigma^2/\left(1-a^2\right)$.

- The evolution of the distribution of x_t

Given our results on the evolution of the mean and the variance of x_t and the fact that we assumed ε_t to be normally distributed, we know that x_t is normally distributed at each point in time. Hence, in order to understand the evolution of the distribution of x_t, we only have to find out how the expected value and variance of the variable x_t evolves. As we have computed just this above, we can express the density for x_t in closed form by (compare ch. 7.2.2)

$$f(x_t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{1}{2}\left(\frac{x_t-\mu_t}{\sigma_t}\right)^2},$$

where the mean μ_t and the variance σ_t^2 are functions of time. These functions are given by $\mu_t = \tilde{x}_t$ from (7.4.2) and $\sigma_t^2 = Var(x_t)$ from (7.4.5). For some illustrating parameter values, we can then draw the evolution of the distribution of x as in the following figure.

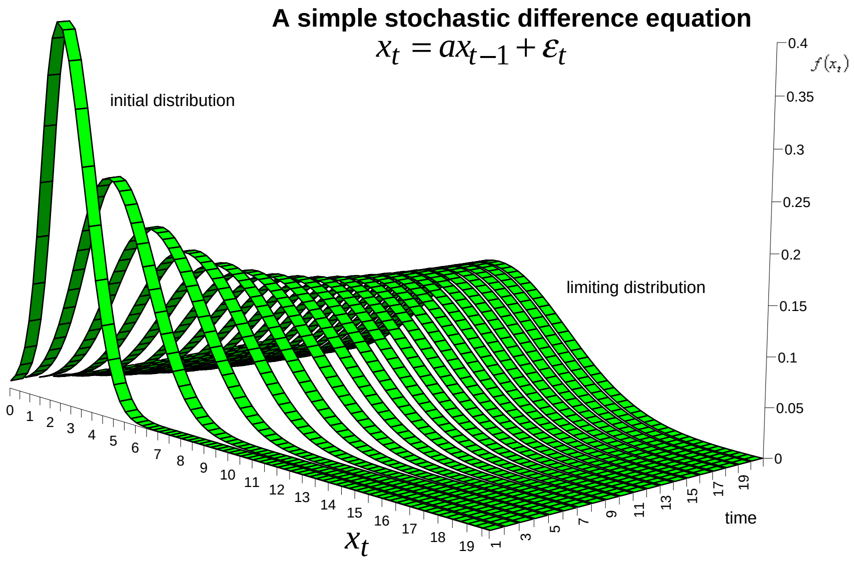


Figure 7.4.1 *Evolution of a distribution over time*

Remember that we were able to plot a distribution for each t only because of properties of the normal distribution. If we had assumed that ε_t is lognormally or equally distributed, we would not have been able to say something about the distribution of x_t easily. The means and variances in (7.4.4) and (7.4.5) would still have been valid but the distribution of x_t for future t is generally unknown for distributions of ε_t other than the normal distribution.

- An interpretation for many agents

Let us now make an excursion into the world of heterogeneous agents. Imagine the variable x_{it} represents the financial wealth of an individual i. Equation (7.4.1) can then be interpreted to describe the evolution of wealth x_{it} over time,

$$x_{it} = ax_{it-1} + \varepsilon_{it}.$$

Given information in period $t = 0$, it predicts the density function $f(x_{it})$ for wealth in all future periods, given some initial value x_{i0}. Shocks are identically and independently distributed (i.i.d.) over time and across individuals.

Assuming a large number of agents i, more precisely a continuum of agents with mass n, a law of large numbers can be applied: Let all agents start with the same wealth $x_{i0} = x_0$ in $t = 0$. Then the density $f(x_{it})$ for the wealth of any individual i in t equals the realized wealth distribution in t for the continuum of agents with mass n. Put differently: when the probability for an individual to hold wealth between some lower and upper bound in t is given by p, the share of individuals that hold wealth between these two bounds in t is also given by p. Total wealth in t in such a pure idiosyncratic risk setup (i.e. no aggregate uncertainty) is then deterministic (as are all other population shares) and is given by

$$x_t = n \int_{-\infty}^{\infty} x_{it} f(x_{it}) dx_{it} = n\mu_t, \tag{7.4.6}$$

where μ_t is the average wealth over individuals or the expected wealth of any individual i.

The same argument can also be made with a discrete number of agents. The wealth x_{it} of individual i at t is a random variable. Let the probability that x_{it} is smaller than $\bar{x}$ be given by $P(x_{it} \leq \bar{x}) = \bar{p}$. Now assume there are n agents and therefore, at each point in time t, there are n independent random variables x_{it}. Denote by $\bar{n}$ the number of random variables x_{it} that have a realization smaller than $\bar{x}$. The share of random variables that have a realization smaller than $\bar{x}$ is denoted by $\bar{q} \equiv \bar{n}/n$. It is then easy to show that $E\bar{q} = \bar{p}$ for all n and, more importantly, $\lim_{n\to\infty} var(\bar{q}) = 0$. This equals in words the statement based on Judd above: The share in the total population n is equal to the individual probability if the population is large. This share becomes deterministic for large populations.

Here is now a formal proof: Define Y_i as the number of realizations below $\bar{x}$ for one x_{it}, i.e. $Y_i \equiv I(x_{it} \leq \bar{x})$ where I is the indicator function which is 1 if the condition in parentheses holds and 0 if not. Clearly, the probability that $Y_i = 1$ is given by $\bar{p}$, i.e. Y_i is Bernoulli($\bar{p}$) distributed, with $EY_i = \bar{p}$ and var$Y_i = \bar{p}(1-\bar{p})$. (The Y_i are i.i.d. if the x_{it} are i.i.d..) Then the share $\bar{q}$ is given by $\bar{q} = \Sigma_{i=1}^{n} Y_i/n$. Its moments are $E\bar{q} = \bar{p}$ and var$\bar{q} = \bar{p}(1-\bar{p})/n$. Hence, the variance tends to zero for n going to infinity. (More technically, $\bar{q}$ tends to $\bar{p}$ in "quadratic mean" and therefore "in probability". The latter means we have a "weak law of large numbers".)

7.4.2 A more general case

Let us now consider the more general difference equation

$$x_t = \alpha_t x_{t-1} + \varepsilon_t,$$

where the coefficient α is now also stochastic

$$\alpha_t \sim \left(a, \sigma_a^2\right), \quad \varepsilon_t \sim \left(\mu, \sigma_\varepsilon^2\right).$$

Analyzing the properties of this process is pretty complex. What can be easily done, however, is to analyze the evolution of moments. As Vervaat (1979) has shown, the limiting distribution has the following moments

$$Ex^j = \Sigma_{k=0}^{j} \binom{j}{k} E\left(\alpha^k \varepsilon^{j-k}\right) Ex^k.$$

Assuming we are interested in the expected value, we would obtain

$$\begin{aligned} Ex^1 &= \Sigma_{k=0}^{1} \binom{1}{k} E\left(\alpha^k \varepsilon^{1-k}\right) Ex^k = \binom{1}{0} E\left(\alpha^0 \varepsilon^1\right) Ex^0 + \binom{1}{1} E\left(\alpha^1 \varepsilon^0\right) Ex^1 \\ &= E\varepsilon + E\alpha Ex. \end{aligned}$$

Solving for Ex yields $Ex = \frac{E\varepsilon}{1-E\alpha} = \frac{\mu}{1-a}$.

Chapter 8

Two-period models

8.1 An overlapping generations model

Let us now return to maximization issues. We do so in the context of the most simple example of a dynamic stochastic general equilibrium model. It is a straightforward extension of the deterministic model analyzed in section 2.4.

The structure of the maximization problem of individuals, the timing of when uncertainty reveals itself and what is uncertain depends on the fundamental and exogenous sources of uncertainty. As the fundamental source of uncertainty results here from the technology used by firms, technologies will be presented first. Once this is done, we can derive the properties of the maximization problem households or firms face.

8.1.1 Technology

Let there be an aggregate technology

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{8.1.1}$$

The total factor productivity level A_t is uncertain. We assume that A_t is identically and independently distributed (i.i.d.) in each period. The random variable A_t is positive, has a mean A and a variance σ^2. No further information is needed about its distribution function at this point,

$$A_t \sim \left(A, \sigma^2\right), \quad A_t > 0. \tag{8.1.2}$$

Assuming that total factor productivity is i.i.d. means, inter alia, that there is no technological progress. One can imagine many different distributions for A_t. In principle, all distributions presented in the last section are viable candidates. Hence, we can work with discrete distributions or continuous distributions.

8.1.2 Timing

The sequence of events is as follows. At the beginning of period t, the capital stock K_t is inherited from the last period, given decisions from the last period. Then, total factor productivity is revealed. With this knowledge, firms choose factor employment and households choose consumption (and thereby savings).

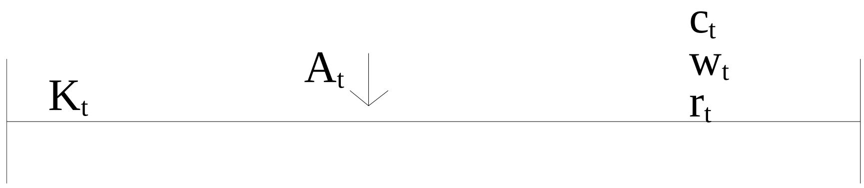


Figure 8.1.1 *Timing of events*

Hence, only "at the end of the day" does one really know how much one has produced. This implies that wages and interest payments are also only known with certainty at the end of the period.

The state of the economy in t is completely described by K_t, L_t and the realization of A_t. All variables of the model are contingent on the state.

8.1.3 Firms

As a consequence of this timing of events, firms do not bear any risk and they pay the marginal product of labour and capital to workers and capital owners at the end of the period,

$$w_t = p_t \frac{\partial Y_t}{\partial L_t}, \tag{8.1.3}$$

$$r_t = p_t \frac{\partial Y_t}{\partial K_t} = p_t A_t \alpha \left(\frac{L_t}{K_t} \right)^{1-\alpha}. \tag{8.1.4}$$

All risk is therefore born by households through labour and capital income.

In what follows, the price will be set to unity, $p_t \equiv 1$. All other prices will therefore be real prices in units of the consumption good.

8.1.4 Intertemporal utility maximization

This is the first time in this book that we encounter a maximization problem with uncertainty. The presentation will therefore be relatively detailed in order to stress crucial features which are new due to the uncertainty.

- General approach

We consider an agent that lives for two periods and works and consumes in a world as just described. Agents consume in both periods and choose consumption such that they maximize expected utility. In all generality concerning the uncertainty, she maximizes

$$\max E_t \left\{ u\left(c_t\right) + \beta u\left(c_{t+1}\right) \right\}, \tag{8.1.5}$$

where β is the subjective discount factor that measures the agent's impatience to consume. The expectations operator has an index t, similar to E_0 in (7.4.3), to indicate that expectations are based on the knowledge concerning random variables which is available in period t. We will see in an instant whether the expectations operator E_t is put at a meaningful place in this objective function. Placing it in front of both instantaneous consumption from c_t and from c_{t+1} is the most general way of handling it. We will also have to specify later what the control variable of the household is.

Imagine the agent chooses consumption for the first and the second period. When consumption is chosen and given wage income w_t, savings adjust such that the budget constraint

$$w_t = c_t + s_t \tag{8.1.6}$$

for the first period holds. Note that this constraint always holds, despite the uncertainty concerning the wage. It holds in realizations, not in expected terms. In the second period, the household receives interest payments on savings made in the first period and uses savings plus interests for consumption,

$$(1 + r_{t+1})\, s_t = c_{t+1}. \tag{8.1.7}$$

One way of solving this problem (for an alternative, see the exercise) is to insert consumption levels from these two constraints into the objective function (8.1.5). This gives

$$\max_{s_t} E_t \left\{ u\left(w_t - s_t\right) + \beta u\left(\left(1 + r_{t+1}\right) s_t\right)\right\}.$$

This nicely shows that the household influences consumption in both periods by choosing savings s_t in the first period. In fact, the only control variable the household can choose is s_t.

Let us now take into consideration, as drawn in the above figure, that consumption takes place at the end of the period after revelation of productivity A_t in that period. Hence, the consumption level in the first period is determined by savings only and is thereby certain. Note that even if consumption c_t (or savings) was chosen *before* revelation of total factor productivity, households would want to consume a different level of consumption c_t after A_t is known. The first choice would therefore be irrelevant and we can therefore focus on consumption choice after revelation of uncertainty right away. The consumption level in the second period is definitely uncertain, however, as the next period interest rate r_{t+1} depends on the realization of A_{t+1} which is unknown in t when decisions about savings s_t are made. The objective function can therefore be rewritten as

$$\max_{s_t} u\left(w_t - s_t\right) + \beta E_t u\left(\left(1 + r_{t+1}\right) s_t\right).$$

For illustration purposes, let us now assume a discrete random variable A_t with a finite number n of possible realizations. Then, this maximization problem can be written as

$$\max_{s_t} \left\{ u\left(w_t - s_t\right) + \beta \Sigma_{i=1}^{n} \pi_i u\left(\left(1 + r_{i,t+1}\right) s_t\right)\right\}$$

where π_i is the probability that the interest rate $r_{i,t+1}$ is in state i in $t+1$. This is the same probability as the probability that the underlying source of uncertainty A_t is in state i. The first-order condition then reads

$$u'(w_t - s_t) = \beta \Sigma_{i=1}^{n} \pi_i u'((1 + r_{i,t+1}) s_t)(1 + r_{i,t+1}) = \beta E_t u'((1 + r_{t+1}) s_t)(1 + r_{t+1}) \tag{8.1.8}$$

Marginal utility of consumption today must equal *expected* marginal utility of consumption tomorrow corrected by interest and time preference rate. Optimal behaviour in an uncertain world therefore means ex ante optimal behaviour, i.e. before random events are revealed. Ex post, i.e. after resolution of uncertainty, behaviour is suboptimal when compared to the case where the realization is known in advance: Marginal utility in t will (with high probability) not equal marginal utility (corrected by interest and time preference rate) in $t+1$. This reflects a simple fact of life: "If I had known before what would happen, I would have behaved differently." Ex ante, behaviour is optimal, ex post, probably not. Clearly, if there was *only one realization for* A_t, i.e. $\pi_1 = 1$ and $\pi_i = 0 \; \forall \; i > 1$, we would have the deterministic first-order condition we had in exercise 1 in ch. 2.

The first-order condition also shows that closed-form solutions are possible if marginal utility is of a multiplicative type. As savings s_t are known, the only quantity which is uncertain is the interest rate r_{t+1}. If the instantaneous utility function $u(.)$ allows us to separate the interest rate from savings, i.e. the argument $(1 + r_{i,t+1}) s_t$ in $u'((1 + r_{i,t+1}) s_t)$ in (8.1.8), an explicit expression for s_t and thereby consumption can be computed. This will be shown in the following example and in exercise 6.

- An example - Cobb-Douglas preferences

Now assume the household maximizes a Cobb-Douglas utility function as in (2.2.1). In contrast to the deterministic setup in (2.2.1), however, expectations about consumption levels need to be formed. Preferences are therefore captured by

$$E_t \{\gamma \ln c_t + (1 - \gamma) \ln c_{t+1}\}. \tag{8.1.9}$$

When we express consumption by savings, we can express the maximization problem by $\max_{s_t} E_t \{\gamma \ln (w_t - s_t) + (1 - \gamma) \ln ((1 + r_{t+1}) s_t)\}$ which is identical to

$$\max_{s_t} \gamma \ln (w_t - s_t) + (1 - \gamma) [\ln s_t + E_t \ln (1 + r_{t+1})]. \tag{8.1.10}$$

The first-order condition with respect to savings reads

$$\frac{\gamma}{w_t - s_t} = \frac{1 - \gamma}{s_t} \tag{8.1.11}$$

and the optimal consumption and saving levels are given by the closed-form solution

$$c_t = \gamma w_t, \qquad c_{t+1} = (1 - \gamma)(1 + r_{t+1}) w_t, \qquad s_t = (1 - \gamma) w_t, \tag{8.1.12}$$

just as in the deterministic case (2.4.7).

Thus, despite the setup with uncertainty, one can compute a closed-form solution of the same structure as in the deterministic solutions (2.2.4) and (2.2.5). What is peculiar here about the solutions and also about the first-order condition is the fact that the expectations operator is no longer visible. One could get the impression that households do not form expectations when computing optimal consumption paths. The expectations operator "got lost" in the first-order condition only because of the logarithm, i.e. the Cobb-Douglas nature of preferences. Nevertheless, there is still uncertainty for an individual being in t : consumption in $t+1$ is unknown as it is a function of the interest-rate in $t+1$.

Exercise 6 shows that closed-form solutions are possible also for the CRRA case beyond Cobb-Douglas.

8.1.5 Aggregation and the reduced form for the CD case

We now aggregate over all individuals. Let there be L newborns each period. Consumption of all young individuals in period t is given from (8.1.12),

$$C_t^y = L\gamma w_t = \gamma(1-\alpha)Y_t.$$

The second equality used competitive wage setting from (8.1.3), the fact that with a Cobb-Douglas technology (8.1.3) can be written as $w_t L_t = p_t (1-\alpha) Y_t$, the normalization of p_t to unity and the identity between number of workers and number of young, $L_t = L$. Note that this expression would hold identically in a deterministic model. With (8.1.4), consumption by old individuals amounts to

$$C_t^o = L[1-\gamma][1+r_t]w_{t-1} = (1-\gamma)(1+r_t)(1-\alpha)Y_{t-1}.$$

Consumption in t depends on output in $t-1$ as savings are based on income in $t-1$.

The capital stock in period $t+1$ is given by savings of the young. One could show this as we did in the deterministic case in ch. 2.4.2. In fact, adding uncertainty would change nothing to the fundamental relationships. We can therefore directly write

$$K_{t+1} = Ls_t = L[1-\gamma]w_t = (1-\gamma)(1-\alpha)Y_t = (1-\gamma)(1-\alpha)A_t K_t^\alpha L^{1-\alpha},$$

where we used the expression for savings for the Cobb-Douglas utility case from (8.1.12). Again, we succeeded in reducing the presentation of the model to one equation in one unknown. This allows us to illustrate the dynamics of the capital stock in this economy by using a simple phase diagram.

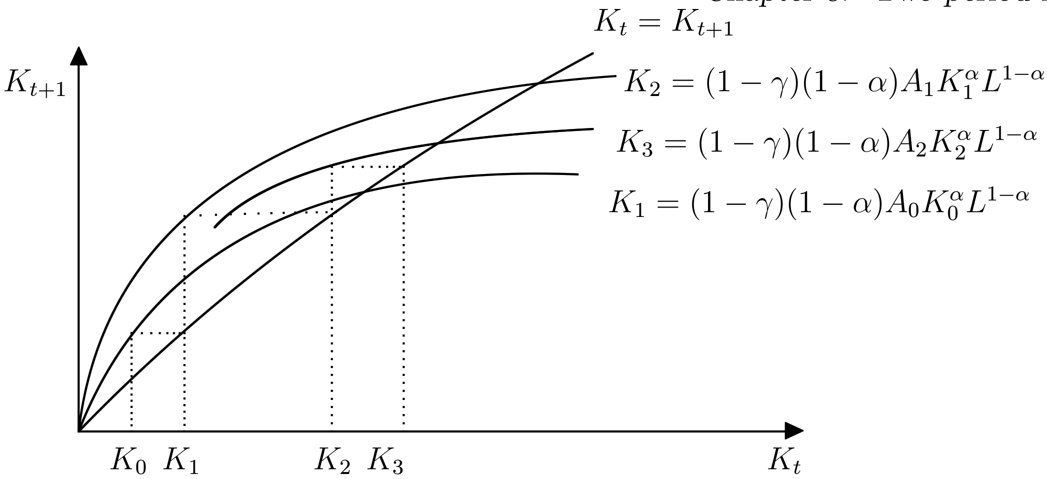


Figure 8.1.2 *Convergence towards a "stochastic steady state"*

In principle this phase diagram looks identical to the deterministic case. There is of course a fundamental difference. The K_{t+1} loci are at a different point in each period. In $t = 0$, K_0 and A_0 are known. Hence, by looking at the K_1 loci, we can compute K_1 as in a deterministic setup. Then, however, TFP changes and, in the example plotted above, A_1 is larger than A_0. Once A_1 is revealed in $t = 1$, the new capital stock for period 2 can be graphically derived. With A_2 revealed in $t = 2$, K_3 can be derived and so on. It is clear that this economy will never end up in one steady state, as in the deterministic case, as A_t is different for each t. As illustrated before in fig. 7.4.1, however, the economy can converge to a unique stationary distribution for K_t.

8.1.6 Some analytical results

- The basic difference equation

In order to better understand the evolution of this economy, let us now look at some analytical results. The logarithm of the capital stock evolves according to

$$\ln K_{t+1} = \ln\left((1-\gamma)(1-\alpha)\right) + \ln A_t + \alpha \ln K_t + (1-\alpha)\ln L_t.$$

Assuming a constant population size $L_t = L$, we can rewrite this equation as

$$\kappa_{t+1} = m_0 + \alpha\kappa_t + \upsilon_t, \qquad \upsilon_t \sim N(\mu_\nu, \sigma_\upsilon^2), \tag{8.1.13}$$

where we used

$$\kappa_t \equiv \ln K_t \tag{8.1.14}$$

$$m_0 \equiv \ln\left[(1-\gamma)(1-\alpha)\right] + (1-\alpha)\ln L \tag{8.1.15}$$

and $\upsilon_t \equiv \ln A_t$ captures the uncertainty stemming from the TFP level A_t. As TFP is i.i.d., so is its logarithm υ_t. Since we assumed the TFP to be lognormally distributed, its logarithm υ_t is normally distributed. When we remove the mean from the random variable by replacing according to $\upsilon_t = \varepsilon_t + \mu_\nu$, where $\varepsilon_t \sim N(0, \sigma_\upsilon^2)$, we obtain $\kappa_{t+1} = m_0 + \mu_\nu + \alpha\kappa_t + \varepsilon_t$.

- The expected value and the variance

We can now compute the expected value of κ_t by following the same steps as ch. 7.4.1, noting that the only additional parameter is m_0. Starting in $t = 0$ with an initial value κ_0, and solving this equation recursively gives

$$\kappa_t = \alpha^t \kappa_0 + (m_0 + \mu_\nu) \frac{1-\alpha^t}{1-\alpha} + \Sigma_{s=0}^{t-1} \alpha^{t-1-s} \varepsilon_s. \tag{8.1.16}$$

Computing the expected value from the perspective of $t = 0$ gives

$$E_0 \kappa_t = \alpha^t \kappa_0 + (m_0 + \mu_\nu) \frac{1-\alpha^t}{1-\alpha}, \tag{8.1.17}$$

where we used $E_0 \varepsilon_s = 0$ for all $s > 0$ (and we set $\varepsilon_0 = 0$ or assumed that expectations in 0 are formed before the realization of ε_0 is known). For t going to infinity, we obtain

$$\lim_{t\to\infty} E_0 \kappa_t = \frac{m_0 + \mu_\nu}{1-\alpha}. \tag{8.1.18}$$

Note that the solution in (8.1.16) is neither difference- nor trend stationary. Only in the limit, we have a (pure) random walk. For a very large t, α^t in (8.1.16) is zero and (8.1.16) implies a random walk,

$$\kappa_t - \kappa_{t-1} = \varepsilon_{t-1}.$$

The variance of κ_t can be computed with (8.1.16), where again independence of all terms is used as in (7.4.5),

$$Var(\kappa_t) = 0 + 0 + Var\left(\Sigma_{s=0}^{t-1} \alpha^{t-1-s} \varepsilon_s\right) = \Sigma_{s=0}^{t-1} \alpha^{2(t-1-s)} \sigma_\nu^2 = \frac{1-\alpha^{2t}}{1-\alpha^2} \sigma_\nu^2 \tag{8.1.19}$$

In the limit, we obtain

$$\lim_{t\to\infty} Var(\kappa_t) = \frac{\sigma_\nu^2}{1-\alpha^2}. \tag{8.1.20}$$

A graphical illustration of these findings would look similar to the one in fig. 7.4.1.

- Relation to fundamental uncertainty

For a more convincing economic interpretation of the mean and the variance, it is useful to express some of the equations, not as a function of properties of the log of A_t, i.e. properties of ν_t, but directly of the level of A_t. As (7.3.4) implies, the mean and variances of these two variables are related in the following fashion

$$\sigma_\nu^2 = \ln\left(1 + \left(\frac{\sigma_A}{\mu_A}\right)^2\right), \tag{8.1.21}$$

$$\mu_\nu = \ln \mu_A - \frac{1}{2}\sigma_\nu^2 = \ln \mu_A - \frac{1}{2} \ln \left(1 + \left(\frac{\sigma_A}{\mu_A} \right)^2 \right). \tag{8.1.22}$$

We can insert these expressions into (8.1.17) and (8.1.19) and study the evolution over time or directly focus on the long-run distribution of $\kappa_t \equiv \ln K_t$. Doing so by inserting into (8.1.18) and (8.1.20) gives

$$\lim_{t\to\infty} E_0 \kappa_t = \frac{m_0 + \ln \mu_A - \frac{1}{2} \ln \left(1 + \left(\frac{\sigma_A}{\mu_A} \right)^2 \right)}{1-\alpha}, \quad \lim_{t\to\infty} Var(\kappa_t) = \frac{\ln \left(1 + \left(\frac{\sigma_A}{\mu_A} \right)^2 \right)}{1-\alpha^2}.$$

These equations tell us that uncertainty in TFP as captured by σ_A not only affects the spread of the long-run distribution of the capital stock but also its mean. More uncertainty leads to a lower long-run mean of the capital stock. This is interesting given the standard result of precautionary saving and the portfolio effect (in setups with more than one asset).

8.1.7 CRRA and CARA utility functions

Before concluding this first analysis of optimal behaviour in an uncertain world, it is useful to explicitly introduce CRRA (constant relative risk aversion) and CARA (constant absolute risk aversion) utility functions. Both are widely used in various applications. The Arrow-Pratt measure of absolute risk aversion is $-u''(c)/u'(c)$ and the measure of relative risk aversion is $-cu''(c)/u'(c)$. An individual with uncertain consumption at a small risk would be willing to give up a certain *absolute* amount of consumption which is proportional to $-u''(c)/u'(c)$ to obtain certain consumption. The *relative* amount she would be willing to give up is proportional to the measure of relative risk aversion.

The CRRA utility function is the same function as the CES utility function which we know from deterministic setups in (2.2.10) and (5.3.2). Inserting a CES utility function $(c^{1-\sigma} - 1)/(1-\sigma)$ into these two measures of risk aversion gives a measure of absolute risk aversion of $-\frac{-\sigma c(\tau)^{-\sigma-1}}{c(\tau)^{-\sigma}} = \sigma/c(\tau)$ and a measure of relative risk aversion of σ, which is minus the inverse of the intertemporal elasticity of substitution. This is why the CES utility function is also called CRRA utility function. Even though this is not consistently done in the literature, it seems more appropriate to use the term CRRA (or CARA) in setups with uncertainty only. In a certain world without risk, risk-aversion plays no role.

The fact that the same parameter σ captures risk aversion and intertemporal elasticity of substitution is not always desirable as two different concepts should be captured by different parameters. The latter can be achieved by using a recursive utility function of the Epstein-Zin type.

The typical example for a utility function with constant absolute risk aversion is the exponential utility function $u(c(\tau)) = -e^{-\sigma c}$ where σ is the measure of absolute risk aversion. Given relatively constant risk-premia over time, the CRRA utility function seems to be preferable for applications.

See "further reading" on references to a more in-depth analysis of these issues.

8.2 Risk-averse and risk-neutral households

Previous analyses in this book have worked with the representative-agent assumption. This means that we neglected all potential differences across individuals and assumed that they are all the same (especially identical preferences, labour income and initial wealth). We now extend the analysis of the last section by (i) allowing individuals to give loans to each other. These loans are given at a riskless endogenous interest rate r. As before, there is a "normal" asset which pays an uncertain return r_{t+1}. We also (ii) assume there are two types of individuals, the risk-neutral ones and the risk-averse, denoted by $i = a, n$. The second assumption is crucial: if individuals were identical, i.e. if they had identical preferences and experienced the same income streams, no loans would be given in equilibrium. In this world with heterogeneous agents, we want to understand who owns which assets. We keep the analysis simple by analyzing a partial equilibrium setup.

- Households

The budget constraints (8.1.6) and (8.1.7) of all individuals i now read

$$w_t = c_t^i + s_t^i, \tag{8.2.1}$$

$$c_{t+1}^i = s_t^i \left[1 + r^i\right]. \tag{8.2.2}$$

In the first period, there is the classic consumption-savings choice. In addition, there is an investment problem as savings need to be allocated to the two types of assets. Consumption in the second period is paid for entirely by capital income. Interests paid on the portfolio amount to

$$r^i = \theta^i r_{t+1} + \left(1 - \theta^i\right) r. \tag{8.2.3}$$

Here and in subsequent chapters, θ^i denotes the share of wealth held by individual i in the risky asset.

We solve this problem by the substitution method which gives us an unconstrained maximization problem. A household i with time preference rate ρ and therefore discount factor $\beta = 1/(1+\rho)$ maximizes

$$U_t^i = u\left(w_t - s_t^i\right) + \beta E_t u\left(\left[1 + r^i\right] s_t^i\right) \to \max_{s_t^i, \theta^i} \tag{8.2.4}$$

by now choosing two control variables: The amount of resources not used in the first period for consumption, i.e. savings s_t^i, and the share θ^i of savings held in the risky asset.

First-order conditions for s_t^i and θ^i, respectively, are

$$u'\left(c_t^i\right) = \beta E_t \left\{u'\left(c_{t+1}^i\right)\left[1 + r^i\right]\right\}, \tag{8.2.5}$$

$$Eu'\left(c_{t+1}^i\right)\left[r_{t+1} - r\right] = 0. \tag{8.2.6}$$

Note that the first-order condition for consumption (8.2.5) has the same interpretation, once slightly rewritten, as the interpretation in deterministic two-period or infinite horizon models (see (2.2.6) in ch. 2.2.2 and (3.1.6) in ch. 3.1.2). When rewriting it as

$$E_t \left\{ \frac{\beta u'\left(c_{t+1}^i\right)}{u'\left(c_t^i\right)} \frac{1}{1/\left(1+r^i\right)} \right\} = 1, \tag{8.2.7}$$

we see that optimal behaviour again requires us to equate marginal utilities today and tomorrow (the latter in its present value) with relative prices today and tomorrow (the latter also in its present value). Of course, in this stochastic environment, we need to express everything in expected terms. As the interest rates and consumption tomorrow are jointly uncertain, we can not bring it exactly in the form as known from above in (2.2.6) and (3.1.6). However, this will be possible, further below in (9.1.10) in ch. 9.1.3.

The first-order condition for θ says that expected returns from giving a loan and holding the risky asset must be identical. Returns consist of the interest rate times marginal utility. This condition can best be understood when first thinking of a certain environment. In this case, (8.2.6) would read $r_{t+1} = r$: agents would be indifferent when holding two assets only if they receive the same interest rate on both assets. Under uncertainty and with risk-neutrality of agents, i.e. $u' = const.$, we get $E_t r_{t+1} = r$: Agents hold both assets only if the expected return from the risky assets equals the certain return from the riskless asset.

Under risk-aversion, we can write this condition as $E_t u'\left(c_{t+1}^i\right) r_{t+1} = E_t u'\left(c_{t+1}^i\right) r$ (or, given that r is non-stochastic, as $E_t u'\left(c_{t+1}^i\right) r_{t+1} = r E_t u'\left(c_{t+1}^i\right)$). This says that agents do not value the interest rate per se but rather the extra utility gained from holding an asset: An asset provides interest of r_{t+1} which increases utility by $u'\left(c_{t+1}^i\right) r_{t+1}$. The share of wealth held in the risky asset then depends on the increase in utility from realizations of r_{t+1} across various states and the expectations operator computes a weighted sum of theses utility increases: $E_t u'\left(c_{t+1}^i\right) r_{t+1} \equiv \Sigma_{j=1}^n u'\left(c_{jt+1}^i\right) r_{jt+1} \pi_j$, where j is the state, r_{jt+1} the interest rate in this state and π_j the probability for state j to occur. An agent is then indifferent between holding two assets when expected utility-weighted returns are identical.

- Risk-neutral and risk-averse behaviour

For risk-neutral individuals (i.e. the utility function is linear in consumption), the first-order conditions become

$$1 = \beta E\left[1 + \theta^n r_{t+1} + (1-\theta^n) r\right], \tag{8.2.8}$$

$$E r_{t+1} = r. \tag{8.2.9}$$

The first-order condition for how to invest implies, together with (8.2.8), that the endogenous interest rate for loans is pinned down by the time preference rate,

$$1 = \beta\left[1 + r\right] \Leftrightarrow r = \rho \tag{8.2.10}$$

as the discount factor is given by $\beta = (1+\rho)^{-1}$ as stated before in (8.2.4). Reinserting this result into (8.2.9) shows that we need to assume that an interior solution for $Er_{t+1} = \rho$ exists. This is not obvious for an exogenously given distribution for the interest rate r_{t+1} but it is more plausible for a situation where r_t is stochastic but endogenous as in the analysis of the OLG model in the previous chapter.

For risk-averse individuals with $u(c_t^a) = \ln c_t^a$, the first-order condition for consumption reads with i in (8.2.3) being replaced by a

$$\frac{1}{c_t^a} = \beta E \frac{1}{c_{t+1}^a}\left[1 + \theta^a r_{t+1} + (1-\theta^a) r\right] = \beta E \frac{1}{s_t^a} = \beta \frac{1}{s_t^a} \tag{8.2.11}$$

where we used (8.2.2) for the second equality and the fact that s_t as a control variable is deterministic for the third. Hence, as in the last section, we can derive explicit expressions. Use (8.2.11) and (8.2.1) and find

$$s_t^a = \beta c_t^a \Leftrightarrow w_t - c_t^a = \beta c_t^a \Leftrightarrow c_t^a = \frac{1}{1+\beta} w_t.$$

This gives with (8.2.1) again and with (8.2.2)

$$s_t^a = \frac{\beta}{1+\beta} w_t, \qquad c_{t+1}^a = \frac{\beta}{1+\beta}\left[1 + \theta^a r_{t+1} + (1-\theta^a) r\right] w_t. \tag{8.2.12}$$

This is our closed-form solution for risk-averse individuals in our heterogeneous-agent economy.

Let us now look at the investment problem of risk-averse households. The derivative of their objective function is given by the left-hand side of the first-order condition (8.2.6) times β. Expressed for logarithmic utility function, and inserting the optimal consumption result (8.2.12) yields

$$\begin{aligned} \frac{d}{d\theta} U_t^a &= \beta E \frac{1}{c_{t+1}^a}\left[r_{t+1} - r\right] = (1+\beta) E \frac{r_{t+1} - r}{1 + \theta^a r_{t+1} + (1-\theta^a) r} \\ &= (1+\beta) E \frac{r_{t+1} - r}{1 + \theta^a (r_{t+1} - r) + r} \equiv (1+\beta) E \frac{X}{\phi + \theta^a X}. \end{aligned} \tag{8.2.13}$$

The last step defined $X \equiv r_{t+1} - r$ as a RV and ϕ as a constant.

- Who owns what?

It can now easily be shown that (8.2.13) implies that risk-averse individuals will not allocate any of their savings to the risky asset, i.e. $\theta^a = 0$. First, observe that the derivative of the expression $EX/(\phi + \theta^a X)$ from (8.2.13) with respect to θ^a is negative

$$\frac{d}{d\theta^a} E \frac{X}{\phi + \theta^a X} = E\left(-\frac{X^2}{(\phi + \theta^a X)^2}\right) < 0 \quad \forall \theta^a.$$

The sign of this derivative can also easily be seen from (8.2.13) as an increase in θ^a implies a larger denominator. Hence, when plotted, the first-order condition is downward sloping in θ^a. Second, by guessing, we find that, with (8.2.9), $\theta^a = 0$ satisfies the first-order condition for investment,

$$\theta^a = 0 \Rightarrow E\frac{X}{\phi + \theta^a X} = EX = 0.$$

Hence, the first-order condition is zero for $\theta^a = 0$. Finally, as the first-order condition is monotonically decreasing, $\theta^a = 0$ is the only value for which it is zero,

$$E\frac{X}{\phi + \theta^a X} = 0 \Leftrightarrow \theta^a = 0.$$

This is illustrated in the following figure.

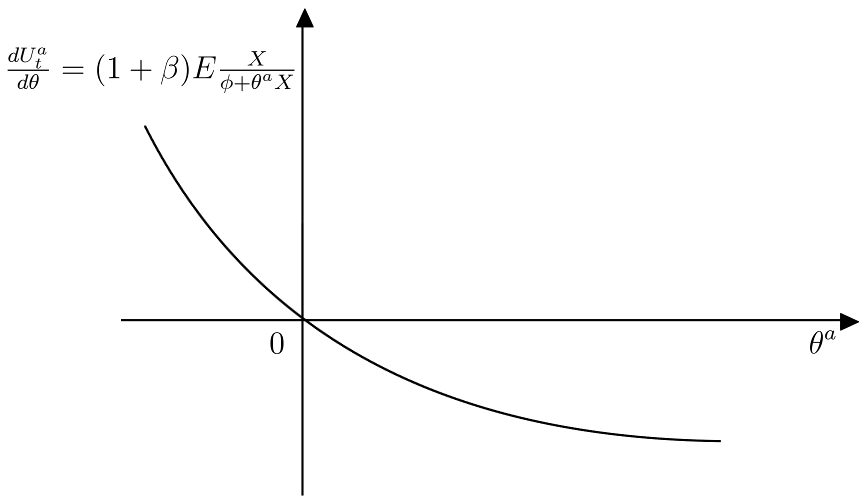


Figure 8.2.1 *The first-order condition (8.2.13) for the share θ^a of savings held in the risky asset*

The figure shows that expected utility of a risk-averse individual increases for negative θ^a and falls for positive θ^a. Risk-averse individuals will hence allocate all of their savings to loans, i.e. $\theta^a = 0$. They give loans to risk-neutral individuals who in turn pay a certain interest rate equal to the expected interest rate. All risk is born by risk-neutral individuals.

8.3 Pricing of contingent claims and assets

8.3.1 The value of an asset

The question is what an individual is willing to pay in $t = 0$ for an asset that is worth p_j (which is uncertain) in $t = 1$. As the assumption underlying the pricing of assets is that individuals behave optimally, the answer is found by considering a maximization problem where individuals solve a saving and portfolio problem.

- The individual

Let an individual's preferences be given by

$$E\left\{u\left(c_0\right)+\beta u\left(c_1\right)\right\}.$$

This individual can invest in a risky and in a riskless investment form. It earns labour income w in the first period and does not have any other source of income. This income w is used for consumption and savings. Savings are allocated to several risky assets j and a riskless bond. The first period budget constraint therefore reads

$$w = c_0 + \Sigma_{j=1}^{n} m_j p_{j0} + b$$

where the number of shares j are denoted by m_j, their price by p_{j0} and the amount of wealth held in the riskless asset is b. The individual consumes all income in the second period which implies the budget constraint

$$c_1 = \Sigma_{j=1}^{n} m_j p_{j1} + (1+r)\, b,$$

where r is the interest rate on the riskless bond.

We can summarize this maximization problem as

$$\max_{c_0, m_j} u\left(c_0\right) + \beta E u\left((1+r)\left(w - c_0 - \Sigma_j m_j p_{j0}\right) + \Sigma_j m_j p_{j1}\right).$$

This expression illustrates that consumption in period one is given by capital income from the riskless interest rate r plus income from assets j. It also shows that there is uncertainty only with respect to period one consumption. This is because consumption in period zero is a control variable. The consumption in period one is uncertain as prices of assets are uncertain. In addition to consumption c_0, the number m_j of shares bought for each asset are also control variables.

- First-order conditions and asset pricing

Computing first-order conditions gives

$$u'\left(c_0\right) = (1+r)\,\beta E u'\left(c_1\right)$$

This is the standard first-order condition for consumption. The first-order condition for the number of assets to buy reads for any asset j

$$\beta E\left[u'(c_1)\left((1+r)(-p_{j0})+p_{j1}\right)\right]=0 \Leftrightarrow E\left[u'(c_1)\,p_{j1}\right]=(1+r)\,p_{j0}Eu'(c_1).$$

The last step used the fact that the interest rate and p_{j0} are known in 0 and can therefore be pulled out of the expectations operator on the right-hand side. Reformulating this condition gives

$$p_{j0}=\frac{1}{1+r}E\frac{u'(c_1)}{Eu'(c_1)}p_{j1} \tag{8.3.1}$$

which is an expression that can be given an interpretation as a pricing equation. The price of an asset j in period zero is given by the discounted expected price in period one. The price in period one is weighted by marginal utilities.

8.3.2 The value of a contingent claim

What we are now interested in are prices of contingent claims. Generally speaking, contingent claims are claims that can be made only if certain outcomes occur. The simplest example of the contingent claim is an option. An option gives the buyer the right to buy or sell an asset at a set price on or before a given date. We will consider current contingent claims whose values can be expressed as a function of the price of the underlying asset. Denote by $g(p_{j1})$ the value of the claim as a function of the price p_{j1} of asset j in period one. The price of the claim in zero is denoted by $v(p_{j0})$.

When we add an additional asset, i.e. the contingent claim under consideration, to the set of assets considered in the previous section, then optimal behaviour of households implies

$$v(p_{j0})=\frac{1}{1+r}E\frac{u'(c_1)}{Eu'(c_1)}g(p_{j1}). \tag{8.3.2}$$

If all agents were risk neutral, we would know that the price of such a claim would be given by

$$v(p_{j0})=\frac{1}{1+r}Eg(p_{j1}) \tag{8.3.3}$$

Both equations directly follow from the first-order condition for assets. Under risk neutrality, a household's utility function is linear in consumption and the first derivative is therefore a constant. Including the contingent claim in the set of assets available to households, exactly the same first-order condition for their pricing would hold.

8.3.3 Risk-neutral valuation

There is a strand in the finance literature, see e.g. Brennan (1979), that asks under what conditions a risk neutral valuation of contingent claims holds (risk neutral valuation relationship, RNVR) even when households are risk averse. This is identical to asking under which conditions marginal utilities in (8.3.1) do not show up. We will now briefly illustrate this approach and drop the index j.

Assume the distribution of the price p_1 can be characterized by a density $f(p_1, \mu)$, where $\mu \equiv Ep_1$. Then, if a risk neutral valuation relationship exists, the price of the contingent claim in zero is given by

$$v(p_0) = \frac{1}{1+r}\int g(p_1) f(p_1, \mu) dp_1, \quad \text{with} \quad \mu = (1+r) p_0.$$

This is (8.3.3) with the expectations operator being replaced by the integral over realizations $g(p_1)$ times the density $f(p_1)$. With risk averse households, the pricing relationship would read, under this distribution,

$$v(p_0) = \frac{1}{1+r}\int \frac{u'(c_1)}{Eu'(c_1)} g(p_1) f(p_1, \mu) dp_1.$$

This is (8.3.2) expressed without the expectations operator. The expected value μ is left unspecified here as it is not a priori clear whether this expected value equals $(1+r) p_0$ also under risk aversion. It is then easy to see that a RNVR holds if $u'(c_1)/Eu'(c_1) = f(p_1)/f(p_1, \mu)$. Similar conditions are derived in that paper for other distributions and for longer time horizons.

8.4 Natural volatility I

Natural volatility is a view of why growing economies experience phases of high and phases of low growth. The central belief is that both long-run growth and short-run fluctuations are jointly determined by economic forces that are inherent to any real world economy. Long-run growth and short-run fluctuations are both endogenous and two sides of the same coin: They both stem from the introduction of new technologies.

It is important to note that no exogenous shocks occur according to this approach. In this sense, it differs from real business cycle (RBC) and sunspot models and also from endogenous growth models with exogenous disturbances.

There are various models that analyse this view in more detail and an overview is provided at http://www.waelde.com/nv.html. This section will look at a simple model that provides the basic intuition. More on natural volatility will follow in ch. 11.5.

8.4.1 The basic idea

The basic mechanism of the natural volatility literature (and this is probably a necessary property of any model that wants to explain both short-run fluctuations and long-run growth) is that some measure of productivity (this could be labour or total factor productivity) does not grow smoothly over time as in most models of exogenous or endogenous long-run growth but that productivity follows a step function.

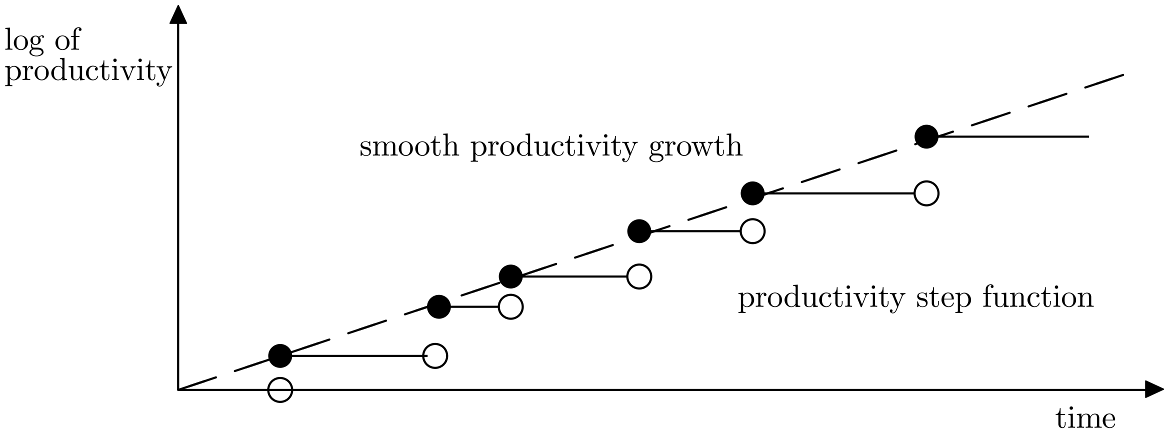


Figure 8.4.1 *Smooth productivity growth in balanced growth models (dashed line) and step-wise productivity growth in models of natural volatility*

With time on the horizontal and the log of productivity on the vertical axis, this figure shows a smooth productivity growth path as the dashed line. This is the smooth growth path that induces balanced growth. In models of natural volatility, the growth path of productivity has periods of no change at all and points in time of discrete jumps. After a discrete jump, returns on investment go up and an upward jump in growth rates results. Growth rates gradually fall over time as long as productivity remains constant. With the next jump, growth rates jump up again. While this step function implies long-run growth as productivity on average grows over time, it also implies short-run fluctuations.

The precise economic reasons given for this step function - which is not simply imposed but always follows from some deeper mechanisms - differ from one approach to the other. A crucial implication of this step-function is the implicit belief that economically relevant technological jumps take place once every 4-5 years. Each cycle of an economy and therefore also long-run growth go back to relatively rare events. Fluctuations in time series that are of higher frequency than these 4-5 years either go back to exogenous shocks, to measurement error or other disturbances in the economy.

The step function sometimes captures jumps in total factor productivity, sometimes only in labour productivity for the most recent vintage of a technology. This difference is important for the economic plausibility of the models. Clearly, one should not build a theory on large aggregate shocks to TFP, as those are not easily observable. Some papers in the literature show indeed how small changes in technology can have large effects (see "further reading").

This section presents the simple stochastic natural volatility model which allows us to show the difference from exogenous shock models in the business cycle most easily .

8.4.2 A simple stochastic model

This section presents the simplest possible model that allows us to understand the difference between the stochastic natural volatility and the RBC approach.

- Technologies

Let the technology be described by a Cobb-Douglas specification,

$$Y_t = A_t K_t^{\alpha} L^{1-\alpha},$$

where A_t represents total factor productivity, K_t is the capital stock and L are hours worked. As variations in hours worked are not required for the main argument here, we consider L constant. Capital can be accumulated according to

$$K_{t+1} = (1-\delta) K_t + I_t$$

Total factor productivity follows

$$A_{t+1} = (1+q_t) A_t \tag{8.4.1}$$

where

$$q_t = \left\{ \begin{array}{c} \bar{q} \\ 0 \end{array} \right\} \text{ with probability } \left\{ \begin{array}{c} p_t \\ 1-p_t \end{array} \right. . \tag{8.4.2}$$

The probability depends on resources R_t invested into R&D,

$$p_t = p(R_t). \tag{8.4.3}$$

Clearly, the function $p(R_t)$ in this discrete time setup must be such that $0 \leq p(R_t) \leq 1$.

The specification of technological progress in (8.4.2) is probably best suited to point out the differences to RBC type approaches: The probability that a new technology occurs is endogenous. This shows both the "new growth literature" tradition and the differences from endogenous growth type RBC models. In the latter approach, the growth rate is endogenous but shocks are still exogenously imposed. Here, the source of growth and fluctuations all stem from one and the same source, the jumps in q_t in (8.4.2).

- Optimal behaviour by households

The resource constraint the economy needs to obey in each period is given by

$$C_t + I_t + R_t = Y_t,$$

where C_t, I_t and R_t are aggregate consumption, investment and R&D expenditure, respectively. Assume that optimal behaviour by households implies consumption and investment into R&D amounting to

$$C_t = s_C Y_t, \quad R_t = s_R Y_t,$$

where s_C is the consumption and s_R the saving rate in R&D. Both of them are constant. This would be the outcome of a two-period maximization problem or an infinite horizon maximization problem with some parameter restriction. As the natural volatility literature has various papers where the saving rate is not constant, it seems reasonable not to develop an optimal saving approach here fully as it is not central to the natural volatility view. See "further reading", however, for references to papers which develop a full intertemporal approach.

8.4.3 Equilibrium

Equilibrium is determined by

$$\begin{aligned} K_{t+1} &= (1-\delta)\, K_t + Y_t - R_t - C_t \\ &= (1-\delta)\, K_t + (1 - s_R - s_C)\, Y_t. \end{aligned}$$

plus the random realization of technology jumps, where the probability of a jump depends on investment in R&D,

$$p_t = p\,(s_R Y_t)\,.$$

Assume we start with a technological level of A_0. Let there be no technological progress for a while, i.e. $q_t = 0$ for a certain number of periods t. Then the capital stock converges to its "temporary steady state" defined by $K_{t+1} = K_t \equiv K$,

$$\delta K = (1 - s_R - s_C)\, A_0 K^{\alpha} L^{1-\alpha} \Leftrightarrow K^{1-\alpha} = \frac{1 - s_R - s_C}{\delta} A_0 L^{1-\alpha}. \tag{8.4.4}$$

In a steady state, all variables are constant over time. Here, variables are constant only temporarily until the next technology jump occurs. This is why the steady state is said to be temporary. The convergence behaviour to the temporary steady state is illustrated in the following figure.

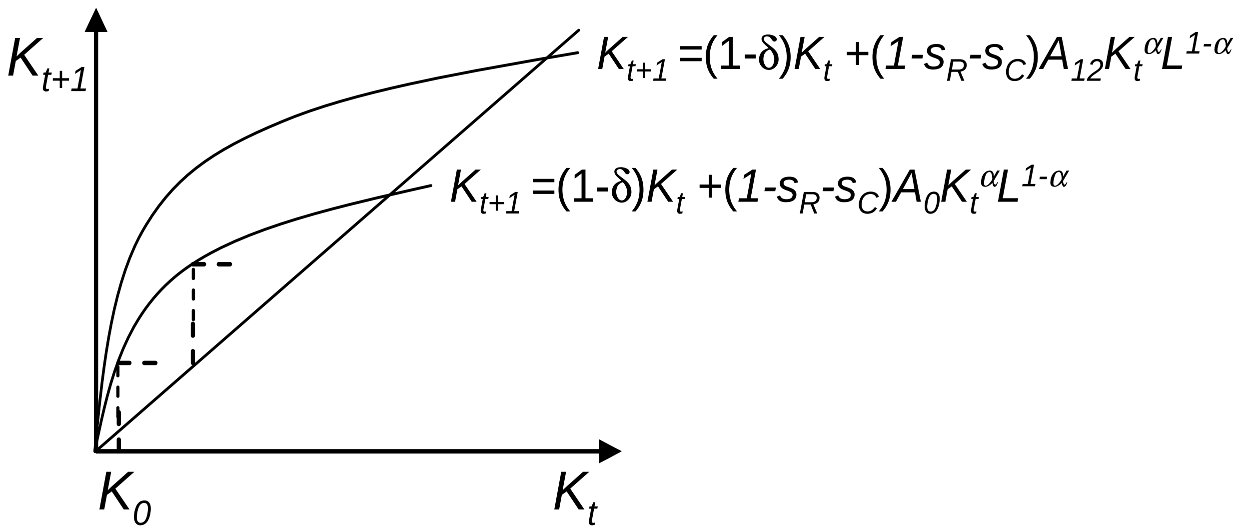


Figure 8.4.2 *The Sisyphus economy - convergence to a temporary steady state*

With a new technology coming in, say, period 12, the total factor productivity increases, according to (8.4.1), from A_0 to $A_{12} = (1 + \bar{q})\, A_0$. The K_{t+1} line shifts upwards, as shown in fig. 8.4.2. As a consequence, which can also be seen in (8.4.4), the steady state increases. Subsequently, the economy approaches this new steady state. As growth is higher immediately after a new technology is introduced (growth is high when the economy is far away from its steady state), the growth rate after the introduction of a new technology is high and then gradually falls. Cyclical growth is therefore characterized by a Sisyphus-type behaviour: K_t permanently approaches the current, temporary steady state. Every now and then, however, this steady state jumps outwards and capital starts approaching it again.

8.5 Further reading and exercises

More on basic concepts of random variables than in ch. 7.1 and 7.2 can be found in Evans, Hastings and Peacock (2000). A very useful reference is Spanos (1999) who also treats functions of several random variables or Severini (2005). For a more advanced treatment, see Johnson, Kotz and Balakrishnan (1995). The results on distributions here are used and extended in Bossmann, Kleiber and Wälde (2007).

There is a long discussion on the application of laws of large numbers in economics. An early contribution is by Judd (1985). See Kleiber and Kotz (2003) for more on distributions.

The example for the value of an asset is inspired by Brennan (1979).

The literature on natural volatility can be found in ch. 11 on p. 293.

See any good textbook on Micro or Public economics (e.g. Mas-Colell et al., 1995, Atkinson and Stiglitz, 1980) for a more detailed treatment of measures of risk aversion. There are many papers using a CARA utility function. Examples include Hassler et al. (2005), Acemoglu and Shimer (1999) and Shimer and Werning (2007, 2008).

Epstein-Zin preferences have been developed by Epstein and Zin (1989) for discrete time setups and applied inter alia in Epstein and Zin (1991). For an application in macroeconomics in discrete time building on the Kreps-Porteus approach, see Weil (1990). The continuous time representation was developed in Svensson (1989). See also Obstfeld (1994) or Epaulard and Pommeret (2003).

Exercises Chapter 7 and 8 Applied Intertemporal Optimization

Stochastic difference equations and applications

1. Properties of random variables
   Is the following function a density function? Draw this function.
   $$f(x) = a\lambda e^{-|\lambda x|}, \qquad \lambda > 0. \tag{8.5.1}$$
   State possible assumptions about the range of x and values for a.

2. The exponential distribution
   Assume the time between two rare events (e.g. the time between two earthquakes or two bankruptcies of a firm) is exponentially distributed.
   (a) What is the distribution function of an exponential distribution?
   (b) What is the probability of no event between 0 and $\bar{x}$?
   (c) What is the probability of at least one event between 0 and $\bar{x}$?

3. The properties of uncertain technological change
   Assume that total factor productivity in (8.1.2) is given by
   (a)
   $$A_t = \left\{ \begin{array}{c} \bar{A} \text{ with p} \\ \underline{A} \text{ with 1-p} \end{array} \right\},$$
   (b) the density function
   $$f(A_t), \quad A_t \in \left[\underline{A}, \bar{A}\right[,$$
   (c) the probability function $g(A_t), \quad A_t \in \{A_1, A_2, ..., A_n\}$,
   (d) $A_t = A_i$ with probability p_i and
   (e) $\ln A_{t+1} = \gamma \ln A_t + \varepsilon_{t+1}$. Make assumptions for γ and ε_{t+1} and discuss them.

   What is the expected total factor productivity and what is its variance? What is the expected output level and what is its variance, given a technology as in (8.1.1)?

4. Stochastic difference equations
Consider the following stochastic difference equation

$$y_{t+1} = by_t + m + v_{t+1}, \quad v_t \sim N(0, \sigma_v^2)$$

(a) Describe the limiting distribution of y_t.

(b) Does the expected value of y_t converge to its fixpoint monotonically? How does the variance of y_t evolve over time?

5. Saving under uncertainty
Consider an individual's maximization problem

$$\max E_t \{u(c_t) + \beta u(c_{t+1})\}$$
$$\text{subject to } w_t = c_t + s_t, \quad (1 + r_{t+1}) s_t = c_{t+1}$$

(a) Solve this problem by replacing her second period consumption by an expression that depends on first period consumption.

(b) Consider now the individual's decision problem given the utility function $u(c_t) = c_t^\sigma$. Should you assume a parameter restriction on σ?

(c) What is your implicit assumption about β? Can it be negative or larger than one? Can the time preference rate ρ, where $\beta = (1+\rho)^{-1}$, be negative?

6. Closed-form solution for a CRRA utility function
Let households maximize a CRRA-utility function

$$U_t = E_t \left[\gamma c_t^{1-\sigma} + (1-\gamma) c_{t+1}^{1-\sigma}\right] = \gamma c_t^{1-\sigma} + (1-\gamma) E_t c_{t+1}^{1-\sigma}$$

subject to budget constraints (8.1.6) and (8.1.7).

(a) Show that an optimal consumption-saving decision given budget constraints (8.1.6) and (8.1.7) implies savings of

$$s_t = \frac{w_t}{1 + \left\{\frac{\gamma}{(1-\gamma)\Phi}\right\}^\varepsilon},$$

where $\varepsilon = 1/\sigma$ is the intertemporal elasticity of substitution and $\Phi \equiv E_t\left[(1+r_{t+1})^{1-\sigma}\right]$ is the expected (transformed) interest rate. Show further that consumption when old is

$$c_{t+1} = (1 + r_{t+1}) \frac{w_t}{1 + \left\{\frac{\gamma}{(1-\gamma)\Phi}\right\}^\varepsilon}$$

and that consumption of the young is

$$c_t = \frac{\left\{\frac{\gamma}{(1-\gamma)\Phi}\right\}^\varepsilon}{1 + \left\{\frac{\gamma}{(1-\gamma)\Phi}\right\}^\varepsilon} w_t. \tag{8.5.2}$$

(b) Discuss the link between the savings expression s_t and the one for the logarithmic case in (8.1.12). Point out why c_{t+1} is uncertain from the perspective of t. Is c_t uncertain from the perspective of t - and of $t-1$?

7. OLG in general equilibrium
Build an OLG model in general equilibrium with capital accumulation and autoregressive total factor productivity, $\ln A_{t+1} = \gamma \ln A_t + \varepsilon_{t+1}$ with $\varepsilon_{t+1} \sim N(\varepsilon, \sigma^2)$. What is the reduced form? Can a phase-diagram be drawn?

8. Asset pricing
Under what conditions is there a risk neutral valuation formula for assets? In other words, under which conditions does the following equation hold?

$$p_{jo} = \frac{1}{1+r} E p_{j1}$$

Chapter 9

Multi-period models

Uncertainty in dynamic models is probably most often used in discrete time models. Looking at time with a discrete perspective has the advantage that timing issues are very intuitive: Something happens today, something tomorrow, something the day before. Time in the real world, however, is continuous. Take any two points in time, you will always find a point in time in between. What is more, working with continuous time models under uncertainty has quite some analytical advantages which make certain insights - after an initial heavier investment into techniques - much simpler. We will nevertheless follow the previous structure of this book and first present models in discrete time.

9.1 Intertemporal utility maximization

Maximization problems in discrete times can be solved by using many methods. One particularly useful one is - again - dynamic programming. We therefore start with this method and consider the stochastic sibling to the deterministic case in ch. 3.3.

9.1.1 The setup with a general budget constraint

Due to uncertainty, our objective function is slightly changed. In contrast to (3.1.1), it now reads

$$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau), \tag{9.1.1}$$

where the only difference lies in the expectations operator E_t. As we are in an environment with uncertainty, we do not know all consumption levels c_τ with certainty. We therefore need to form expectations about the implied instantaneous utility from consumption, $u(c_\tau)$. The budget constraint is given by

$$x_{t+1} = f(x_t, c_t, \varepsilon_t). \tag{9.1.2}$$

This constraint shows why we need to form expectations about future utility: The value of the state variable x_{t+1} depends on some random source, denoted by ε_t. Think of this ε_t as uncertain TFP or uncertain returns on investment.

9.1.2 Solving by dynamic programming

The optimal program is defined in analogy to (3.3.2) by

$$V(x_t, \varepsilon_t) \equiv \max_{\{c_\tau\}} U_t \text{ subject to } x_{t+1} = f(x_{t,} c_t, \varepsilon_t).$$

The additional term in the value function is ε_t. It is useful to treat ε_t explicitly as a state variable for reasons we will soon see. Note that this issue is related to the discussion of "what is a state variable?" in ch. 3.4.2. In order to solve the maximization problem, we again follow the three step scheme.

- DP1: Bellman equation and first-order conditions

The Bellman equation is

$$V(x_t, \varepsilon_t) = \max_{c_t} \{u(c_t) + \beta E_t V(x_{t+1}, \varepsilon_{t+1})\}.$$

It again exploits the fact that the objective function (9.1.1) is additively separable, despite the expectations operator, assumes optimal behaviour as of tomorrow and shifts the expectations operator behind instantaneous utility of today as $u(c_t)$ is certain given that c_t is a control variable. The first-order condition is

$$u'(c_t) + \beta E_t \left[V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1}) \frac{\partial x_{t+1}}{\partial c_t} \right] = u'(c_t) + \beta E_t \left[V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1}) \frac{\partial f(.)}{\partial c_t} \right] = 0. \tag{9.1.3}$$

It corresponds to the first-order condition (3.3.5) in a deterministic setup. The "only" difference lies in the expectations operator E_t. Marginal utility from consumption does not equal the loss in overall utility due to less wealth but the expected loss in overall utility. Equation (9.1.3) provides again an (implicit) functional relationship between consumption and the state variable, $c_t = c(x_t, \varepsilon_t)$. As we know all state variables in t, we know the optimal choice of our control variable. Be aware that the expectations operator applies to all terms in the brackets. If no confusion arises, these brackets will be omitted in what follows.

- DP2: Evolution of the costate variable

The second step under uncertainty also starts from the maximized Bellman equation. The derivative of the maximized Bellman equation with respect to the state variable is (using the envelope theorem)

$$V_{x_t}(x_t, \varepsilon_t) = \beta E_t V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1}) \frac{\partial x_{t+1}}{\partial x_t}.$$

Observe that x_{t+1} by the constraint (9.1.2) is given by $f(x_t, c_t, \varepsilon_t)$, i.e. by quantities that are known in t. Hence, the derivative $\partial x_{t+1}/\partial x_t = f_{x_t}$ is non-stochastic and we can write this expression as (note the similarity to (3.3.6))

$$V_{x_t}(x_t, \varepsilon_t) = \beta f_{x_t} E_t V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1}). \tag{9.1.4}$$

This step clearly shows why it is useful to include ε_t as a state variable into the arguments of the value function. If we had not done so, one could get the impression, for the same argument that x_{t+1} is known in t due to (9.1.2), that the shadow price $V_{x_t}(x_{t+1}, \varepsilon_{t+1})$ is non-stochastic as well and the expectations operator would not be needed. Given that the value of the optimal program depends on ε_t, however, it is clear that the costate variable $V'(x_{t+1}, \varepsilon_{t+1})$ is random in t indeed. (Some of the subsequent applications will treat ε_t as an implicit state variable and will not always be as explicit as here.)

- DP3: Inserting first-order conditions

Given that $\partial f(.)/\partial c_t = f_{c_t}$ is non-random in t, we can rewrite the first-order condition as $u'(c_t) + \beta f_{c_t} E_t V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1}) = 0$. Inserting it into (9.1.4) gives $V_{x_t}(x_t, \varepsilon_t) = -\frac{f_{x_t}}{f_{c_t}} u'(c_t)$. Shifting this expression by one period yields $V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1}) = -\frac{f_{x_{t+1}}}{f_{c_{t+1}}} u'(c_{t+1})$. Inserting $V_{x_t}(x_t, \varepsilon_t)$ and $V_{x_{t+1}}(x_{t+1}, \varepsilon_{t+1})$ into the costate equation (9.1.4) again, we obtain

$$u'(c_t) = \beta E_t \frac{f_{c_t}}{f_{c_{t+1}}} f_{x_{t+1}} u'(c_{t+1}).$$

9.1.3 The setup with a household budget constraint

Let us now look at a first example - a household that maximizes utility. The objective function is given by (9.1.1),

$$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau).$$

It is maximized subject to the following budget constraint (of which we know from (2.5.13) or (3.6.6) that it fits well into a general equilibrium setup),

$$a_{t+1} = (1 + r_t) a_t + w_t - p_t c_t. \tag{9.1.5}$$

Note that the budget constraint must hold after realization of random variables, not in expected terms. From the perspective of t, all prices (r_t, w_t, p_t) in t are known, prices in $t+1$ are uncertain.

9.1.4 Solving by dynamic programming

- DP1: Bellman equation and first-order conditions

Having understood in the previous general chapter that uncertainty can be explicitly treated in the form of a state variable, we limit our attention to the endogenous state variable a_t here. It will turn out that this keeps notation simpler. The value of optimal behaviour is therefore expressed by $V(a_t)$ and the Bellman equation can be written as

$$V(a_t) = \max_{c_t} \{u(c_t) + \beta E_t V(a_{t+1})\}.$$

The first-order condition for consumption is

$$u'(c_t) + \beta E_t V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial c_t} = u'(c_t) - \beta E_t V'(a_{t+1}) p_t = 0,$$

where the second step computed the derivative by using the budget constraint (9.1.5). Rewriting the first-order condition yields

$$u'(c_t) = \beta p_t E_t V'(a_{t+1}) \tag{9.1.6}$$

as the price p_t is known in t.

- DP2: Evolution of the costate variable

Differentiating the maximized Bellman equation gives (using the envelope theorem) $V'(a_t) = \beta E_t V'(a_{t+1}) \partial a_{t+1}/\partial a_t$. Again using the budget constraint (9.1.5) for the partial derivative, we find

$$V'(a_t) = \beta [1 + r_t] E_t V'(a_{t+1}). \tag{9.1.7}$$

Again, the term $[1 + r_t]$ was put in front of the expectations operator as r_t is known in t. This difference equation describes the evolution of the shadow price of wealth in the case of optimal consumption choices.

- DP3: Inserting first-order conditions

Inserting the first-order condition (9.1.6) gives $V'(a_t) = [1 + r_t] \frac{u'(c_t)}{p_t}$. Inserting this expression into the differentiated maximized Bellman equation (9.1.7) twice gives a nice Euler equation,

$$[1 + r_t] \frac{u'(c_t)}{p_t} = \beta [1 + r_t] E_t [1 + r_{t+1}] \frac{u'(c_{t+1})}{p_{t+1}} \Leftrightarrow \frac{u'(c_t)}{p_t} = E_t \frac{\beta u'(c_{t+1})}{(1 + r_{t+1})^{-1} p_{t+1}}. \tag{9.1.8}$$

Rewriting it as we did before with (8.2.7), we get

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{p_t}{(1 + r_{t+1})^{-1} p_{t+1}} \right\} = 1 \tag{9.1.9}$$

which allows us to give the same interpretation as the first-order condition in a two-period model, both deterministic (as in eq. (2.2.6) in ch. 2.2.2) and stochastic (as in eq. (8.2.5) in ch. 8.2) and as in deterministic infinite horizon models (as in eq. (3.1.6) in ch. 3.1.2): Relative marginal utility must be equal to relative marginal prices - taking into account that marginal utility in $t + 1$ is discounted at β and the price in $t + 1$ is discounted by using the interest rate.

Using two further assumptions, the expression in (9.1.8) can be rewritten such that we come even closer to the deterministic Euler equations: First, let us choose the output good as numéraire and thereby set prices $p_t = p_{t+1} = \bar{p}$ as constant. This allows us to remove p_t and p_{t+1} from (9.1.8). Second, we assume that the interest rate is known (say we are in a small open economy with international capital flows). Hence, expectations are formed only with respect to consumption in $t+1$. Taking these two aspects into account, we can write (9.1.8) as

$$\frac{u'(c_t)}{\beta E_t u'(c_{t+1})} = \frac{1}{1/(1+r_{t+1})}. \tag{9.1.10}$$

This is now as close to (2.2.6) and (3.1.6) as possible. The ratio of (expected discounted) marginal utilities is identical to the ratio of relative prices.

9.2 A central planner

Let us now consider the classic central planner problem for a stochastic growth economy. We specify an optimal growth model where total factor productivity is uncertain. In addition to this, we also allow for oil as an input good for production. This allows us to understand how some variables can easily be substituted out in a maximization problem even though we are in an intertemporal stochastic world.

Consider a technology where output is produced with oil O_t in addition to the standard factors of production,

$$Y_t = A_t K_t^{\alpha} O_t^{\beta} L_t^{1-\alpha-\beta}$$

Again, total factor productivity A_t is stochastic. Now, the price of oil, q_t, is stochastic as well. Let capital evolve according to

$$K_{t+1} = (1-\delta) K_t + Y_t - q_t O_t - C_t \tag{9.2.1}$$

which is a trade balance and good market clearing condition all in one. The central planner maximizes

$$\max_{\{C_\tau, O_\tau\}} E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau)$$

by choosing a path of aggregate consumption flows C_τ and oil consumption O_τ. At t, all variables indexed t are known. The only uncertainty concerns TFP and the price of oil in future periods.

- DP1: Bellman equation and first-order conditions

The Bellman equation reads $V(K_t) = \max_{C_t, O_t} \{u(C_t) + \beta E_t V(K_{t+1})\}$. The only state variable included as argument is the capital stock. Other state variables (like the price q_t of oil) could be included but would not help in the derivation of optimality conditions. See the discussion in ch. 3.4.2. The first-order condition for consumption is

$$u'(C_t) + \beta E_t \frac{dV(K_{t+1})}{dK_{t+1}} [-1] = 0 \Leftrightarrow u'(C_t) = \beta E_t V'(K_{t+1})$$

For oil it reads

$$\beta E_t V'(K_{t+1}) \frac{\partial}{\partial O_t} [Y_t - q_t O_t] = 0 \Leftrightarrow \frac{\partial Y_t}{\partial O_t} = q_t.$$

The last step used the fact that all variables in t are known and the partial derivative with respect to O_t can therefore be moved in front of the expectations operator - which then cancels. We therefore have obtained a standard period-for-period optimality condition as it is known from static problems. This is the typical result for a control variable which has no intertemporal effects as, here, imports of oil affect output Y_t and the costs $q_t O_t$ in the resource constraint (9.2.1) contemporaneously.

- DP2: Evolution of the costate variable

The derivative of the Bellman equation with respect to the capital stock K_t gives (using the envelope theorem)

$$V'(K_t) = \beta E_t \frac{\partial}{\partial K_t} V(K_{t+1}) = \beta \left[1 - \delta + \frac{\partial Y_t}{\partial K_t}\right] E_t V'(K_{t+1}) \tag{9.2.2}$$

just as in the economy without oil. The term $\left[1 - \delta + \frac{\partial Y_t}{\partial K_t}\right]$ can be pulled out of the expectation operator as A_t and thereby $\frac{\partial Y_t}{\partial K_t}$ is known at the moment of the savings decision.

- DP3: Inserting first-order conditions

Following the same steps as above without oil, we again end up with

$$u'(C_t) = \beta E_t u'(C_{t+1}) \left[1 - \delta + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\right]$$

The crucial difference is, that now expectations are formed with respect to technological uncertainty and uncertainty concerning the price of oil.

9.3 Asset pricing in a one-asset economy

This section returns to the question of asset pricing. Ch. 8.3.1 treated this issue in a partial equilibrium setting. Here, we take a general equilibrium approach and use a simple stochastic model with one asset, physical capital. We then derive an equation that expresses the price of capital in terms of income streams from holding capital. In order to be as explicit as possible about the nature of this (real) capital price, we do not choose a numéraire good in this section.

9.3.1 The model

- Technologies

The technology used to produce a homogeneous output good is of the simple Cobb-Douglas form

$$Y_t = A_t K_t^{\alpha} L^{1-\alpha}, \tag{9.3.1}$$

where TFP A_t is stochastic. Labour supply L is exogenous and fix, the capital stock is denoted by K_t.

- Households

Household preferences are standard and given by

$$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau).$$

We start from a budget constraint that can be derived like the budget constraint (2.5.10) in a deterministic world. Wealth is held in units of capital K_t where the price of one unit is v_t. When we define the interest rate as

$$r_t \equiv \frac{w_t^K}{v_t} - \delta, \tag{9.3.2}$$

the budget constraint (2.5.10) reads

$$k_{t+1} = (1 + r_t) k_t + \frac{w_t}{v_t} - \frac{p_t}{v_t} c_t. \tag{9.3.3}$$

- Goods market

Investment and consumption goods are traded on the same goods market. Total supply is given by Y_t, demand is given by gross investment $K_{t+1} - K_t + \delta K_t$ and consumption C_t. Expressed in a well-known way, goods market equilibrium yields the resource constraint of the economy,

$$K_{t+1} = K_t + Y_t - C_t - \delta K_t. \tag{9.3.4}$$

9.3.2 Optimal behaviour

Firms maximize instantaneous profits which implies first-order conditions

$$w_t = p_t \partial Y_t / \partial L, \quad w_t^K = p_t \partial Y_t / \partial K_t. \tag{9.3.5}$$

Factor rewards are given by their value marginal products.

Given the households' preferences and the constraint in (9.3.3), optimal behaviour by households is described by (this follows identical steps as for example in ch. 9.1.3 and is treated in ex. 2),

$$\frac{u'(C_t)}{p_t/v_t} = E_t \frac{\beta u'(C_{t+1})}{(1 + r_{t+1})^{-1} p_{t+1}/v_{t+1}}. \tag{9.3.6}$$

This is the standard Euler equation extended for prices, given that we have not chosen a numéraire. We replaced c_t by C_t to indicate that this is the evolution of aggregate (and not individual) consumption.

9.3.3 The pricing relationship

Let us now turn to the main objective of this section and derive an expression for the real price of one unit of capital, i.e. the price of capital in units of the consumption good. Starting from the Euler equation (9.3.6), we insert the interest rate (9.3.2) in its general formulation, i.e. including all prices, and rearrange to find

$$u'(C_t)\frac{v_t}{p_t} = \beta E_t u'(C_{t+1})\left(1+\frac{w^K_{t+1}}{v_{t+1}}-\delta\right)\frac{v_{t+1}}{p_{t+1}} = \beta E_t u'(C_{t+1})\left((1-\delta)\frac{v_{t+1}}{p_{t+1}}+\frac{\partial Y_{t+1}}{\partial K_{t+1}}\right). \tag{9.3.7}$$

Now define a discount factor $\Phi_{t+1} \equiv u'(C_{t+1})/u'(C_t)$ and d_t as the "net dividend payments", i.e. payments to the owner of one unit of capital. Net dividend payments per unit of capital amount to the marginal product of capital $\partial Y_t/\partial K_t$ minus the share δ of capital that depreciates - that goes kaput - each period times the real price v_t/p_t of one unit of capital, $d_t \equiv \frac{\partial Y_t}{\partial K_t} - \delta\frac{v_t}{p_t}$. Inserting this yields

$$\frac{v_t}{p_t} = \beta E_t \Phi_{t+1}\left[d_{t+1}+\frac{v_{t+1}}{p_{t+1}}\right]. \tag{9.3.8}$$

Note that all variables uncertain from the perspective of today in t appear behind the expectations operator.

Now assume for a second that we are in a deterministic world and the economy is in a steady state. Equation (9.3.8) could then be written with $\Phi_{t+1} = 1$ and without the expectations operator as $\frac{v_t}{p_t} = \beta\left[d_{t+1}+\frac{v_{t+1}}{p_{t+1}}\right]$. Solving this linear differential equation forward, starting in v_t and inserting repeatedly gives

$$\begin{aligned}\frac{v_t}{p_t} &= \beta\left[d_{t+1}+\beta\left[d_{t+2}+\beta\left[d_{t+3}+\beta\left[d_{t+4}+\frac{v_{t+4}}{p_{t+4}}\right]\right]\right]\right]\\ &= \beta d_{t+1}+\beta^2 d_{t+2}+\beta^3 d_{t+3}+\beta^4 d_{t+4}+\beta^4\frac{v_{t+4}}{p_{t+4}}.\end{aligned}$$

Continuing to insert, one eventually and obviously ends up with

$$\frac{v_t}{p_t} = \Sigma^T_{s=1}\beta^s d_{t+s} + \beta^T\frac{v_{t+T}}{p_{t+T}}.$$

The price v_t of a unit of capital is equal to the discounted sum of future dividend payments plus its discounted price (once sold) in $t+T$. In an infinite horizon perspective, this becomes

$$\frac{v_t}{p_t} = \Sigma^\infty_{s=1}\beta^s d_{t+s} + \lim_{T\to\infty}\beta^T\frac{v_{t+T}}{p_{t+T}}.$$

In our stochastic setup, we can proceed according to the same principles as in the deterministic world but need to take the expectations operator and the discount factor Φ_t into account. We replace $\frac{v_{t+1}}{p_{t+1}}$ in (9.3.8) by $\beta E_{t+1}\Phi_{t+2}\left[d_{t+2}+\frac{v_{t+2}}{p_{t+2}}\right]$ and then v_{t+2}/p_{t+2}

and so on to find

$$\frac{v_t}{p_t} = \beta E_t \Phi_{t+1} \left[d_{t+1} + \beta E_{t+1} \Phi_{t+2} \left[d_{t+2} + \beta E_{t+2} \Phi_{t+3} \left[d_{t+3} + \frac{v_{t+3}}{p_{t+3}} \right] \right] \right]$$
$$= E_t \left[\beta \Phi_{t+1} d_{t+1} + \beta^2 \Phi_{t+1} \Phi_{t+2} d_{t+2} + \beta^3 \Phi_{t+1} \Phi_{t+2} \Phi_{t+3} d_{t+3} + \beta^3 \Phi_{t+1} \Phi_{t+2} \Phi_{t+3} \frac{v_{t+3}}{p_{t+3}} \right]$$
$$= E_t \Sigma_{s=1}^3 \Delta_s d_{t+s} + E_t \Delta_3 \frac{v_{t+3}}{p_{t+3}}, \tag{9.3.9}$$

where we defined the discount factor to be

$$\Delta_s \equiv \beta^s \Pi_{n=1}^s \Phi_{t+n} = \beta^s \Pi_{n=1}^s \frac{u'(C_{t+n})}{u'(C_{t+n-1})} = \beta^s \frac{u'(C_{t+s})}{u'(C_t)}.$$

The discount factor adjusts discounting by the preference parameter β, by relative marginal consumption and by prices. Obviously, (9.3.9) implies for larger time horizons $\frac{v_t}{p_t} = E_t \Sigma_{s=1}^T \Delta_s d_{t+s} + E_t \Delta_T v_{t+T}$. Again, with an infinite horizon, this reads

$$\frac{v_t}{p_t} = E_t \Sigma_{s=1}^\infty \Delta_s d_{t+s} + \lim_{T \to \infty} E_t \Delta_T v_{t+T}. \tag{9.3.10}$$

The real price v_t/p_t amounts to the discounted sum of future dividend payments d_{t+s}. The discount factor is Δ_s which contains marginal utilities, relative prices and the individual's discount factor β. The term $\lim_{T \to \infty} E_t \Delta_T v_{t+T}$ is a "bubble term" for the price of capital and can usually be set equal to zero. As the derivation has shown, the expression for the price v_t/p_t is "simply" a rewritten version of the Euler equation.

9.3.4 More real results

- The price of capital again

The result on the determinants of the price of capital is useful for economic intuitions and received a lot of attention in the literature. But can we say more about the real price of capital? The answer is yes and it comes from the resource constraint (9.3.4). This constraint can be understood as a goods market clearing condition. The supply of goods Y_t equals demand resulting from gross investment $K_{t+1} - K_t + \delta K_t$ and consumption. The price of one unit of the capital good therefore equals the price of one unit of the consumption and output good, provided that investment takes place, i.e. $I_t > 0$. Hence,

$$v_t = p_t.. \tag{9.3.11}$$

The real price of capital v_t/p_t is just equal to one. Not surprisingly, capital goods and consumption goods are traded on the same market.

- The evolution of consumption (and capital)

When we want to understand what this model tells us about the evolution of consumption, we can look at a modified version of (9.3.6) by inserting the interest rate (9.3.2) with the marginal product of capital from (9.3.5) and the price expression (9.3.11),

$$u'(C_t) = \beta E_t u'(C_{t+1}) \left(1 + \frac{\partial Y_{t+1}}{\partial K_{t+1}} - \delta\right).$$

This is the standard Euler equation (see e.g. (9.1.10)) that predicts how real consumption evolves over time, given the real interest rate and the discount factor β.

Together with (9.3.4), we have a system in two equations that determine C_t and K_t (given appropriate boundary conditions). The price p_t and thereby the value v_t can not be determined (which is of course a consequence of Walras' law). The relative price is trivially unity from (9.3.11), $v_t/p_t = 1$. Hence, the predictions concerning real variables do not change when a numéraire good is not chosen.

- An endowment economy

Many papers work with pure endowment economies. We will look at such an economy here and see how this can be linked to our setup with capital accumulation. Consider an individual that can save in one asset and whose budget constraint is given by (9.3.3). Let this household behave optimally such that optimal consumption follows (9.3.7). Now change the capital accumulation equation (9.3.4) such that - for whatever reasons - K is constant and let also, for simplicity, depreciation be zero, $\delta = 0$. Then, output is given according to (9.3.1) by $Y_t = A_t K^\alpha L^{1-\alpha}$, i.e. it follows some exogenous stochastic process, depending on the realization of A_t. This is the exogenous endowment of the economy for each period t. Further, consumption equals output in each period, $C_t = Y_t$.

Inserting output into the Euler equation (9.3.7) gives

$$u'(Y_t) \frac{v_t}{p_t} = \beta E_t u'(Y_{t+1}) \left((1-\delta) \frac{v_{t+1}}{p_{t+1}} + \frac{\partial Y_{t+1}}{\partial K_{t+1}}\right)$$

The equation shows that in an endowment economy where consumption is exogenously given at each point in time and households save by holding capital (which is constant on the aggregate level), the price v_t/p_t of the asset changes over time such that households *want* to consume optimally the exogenously given amount Y_t. This equation provides a description of the evolution of the price of the asset in an endowment economy. These aspects were analyzed for example by Lucas (1978) and many others.

9.4 Endogenous labour supply

- The maximization problem

The analysis of business cycles is traditionally performed by including an endogenous labour supply decision in the consumption and saving framework we know from ch. 9.1. We will now solve such an extended maximization problem.

The objective function (9.1.1) is extended to reflect that households value leisure. We also allow for an increase in the size of the household. The objective function now reads

$$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, T - l_\tau) n_\tau,$$

where T is the total endowment of this individual with time and l_τ is hours worked in period τ. Total time endowment T is, say, 24 hours or, subtracting time for sleeping and other regular non-work and non-leisure activities, 15 or 16 hours. Consumption per member of the family is given by c_τ and $u(.)$ is instantaneous utility of this member. The number of members in τ is given by n_τ.

The budget constraint of the household is given by

$$\hat{a}_{t+1} = (1 + r_t) \hat{a}_t + n_t l_t w_t - n_t c_t$$

where $\hat{a}_t \equiv n_t a_t$ is household wealth in period t. Letting w_t denote the hourly wage, $n_t l_t w_t$ stands for total labour income of the family, i.e. the product of individual income $l_t w_t$ times the number of family members. Family consumption in t is $n_t c_t$.

- Solving by the Lagrangian

We now solve the maximization problem $\max U_t$ subject to the constraint by choosing individual consumption c_τ and individual labour supply l_τ. We solve this by using a Lagrange function. A solution by dynamic programming would of course also work.

For the Lagrangian we use a Lagrange multiplier λ_τ for the constraint in τ. This makes it different to the Lagrange approach in ch. 3.1.2 where the constraint was an intertemporal budget constraint. It is similar to ch. 3.7 where an infinite number of multipliers is also used. In that chapter, uncertainty is, however, missing. The Lagrangian here reads

$$\mathcal{L} = E_t \Sigma_{\tau=t}^{\infty} \left\{ \beta^{\tau-t} u(c_\tau, T - l_\tau) n_\tau + \lambda_\tau \left[(1 + r_\tau) \hat{a}_\tau + n_\tau l_\tau w_\tau - n_\tau c_\tau - \hat{a}_{\tau+1} \right] \right\}.$$

We first compute the first-order conditions for consumption and hours worked for one point in time s,

$$\mathcal{L}_{c_s} = E_t \left\{ \beta^{s-t} \frac{\partial}{\partial c_s} u(c_s, T - l_s) n_s - \lambda_s n_s \right\} = 0, \tag{9.4.1}$$

$$\mathcal{L}_{l_s} = E_t \left\{ \beta^{s-t} \left[-\frac{\partial}{\partial (T - l_s)} u(c_s, T - l_s) \right] n_s + \lambda_s n_s w_s \right\} = 0. \tag{9.4.2}$$

As discussed in 3.7, we also need to compute the derivative with respect to the state variable. Consistent with the approach in the deterministic setup of ch. 3.8, we compute the derivative with respect to the true state variable of the family, i.e. with respect to $\hat{a}_\tau$. This derivative is

$$\begin{aligned} \mathcal{L}_{\hat{a}_s} &= E_t \left\{ -\lambda_{s-1} + \lambda_s \left[1 + r_s\right] \right\} = 0 \\ &\Leftrightarrow \; E_t \lambda_{s-1} = E_t \left\{ \left(1 + r_s\right) \lambda_s \right\}. \end{aligned} \tag{9.4.3}$$

This corresponds to (3.7.5) in the deterministic case, only that here we have an expectations operator.

- Optimal consumption

As the choice for the consumption level c_s in (9.4.1) will be made in s, we can assume that we have all information in s at our disposal. When we apply expectations E_s, we see that all expectations are made with respect to variables of s. Cancelling n_s, we therefore know that in all periods s we have

$$\beta^{s-t} \frac{\partial}{\partial c_s} u\left(c_s, T - l_s\right) = \lambda_s. \tag{9.4.4}$$

We can now replace λ_s and λ_{s-1} in (9.4.3) by the expressions we get from this equation where λ_s is directly available and λ_{s-1} is obtained by shifting the optimality condition back in time, i.e. by replacing s by $s-1$. We then find

$$\begin{aligned} E_t \left\{ \beta^{s-1-t} \frac{\partial}{\partial c_{s-1}} u\left(c_{s-1}, T - l_{s-1}\right) \right\} &= E_t \left\{ \left(1 + r_s\right) \beta^{s-t} \frac{\partial}{\partial c_s} u\left(c_s, T - l_s\right) \right\} \Leftrightarrow \\ E_t \left\{ \frac{\partial}{\partial c_{s-1}} u\left(c_{s-1}, T - l_{s-1}\right) \right\} &= \beta E_t \left\{ \left(1 + r_s\right) \frac{\partial}{\partial c_s} u\left(c_s, T - l_s\right) \right\}. \end{aligned}$$

Now imagine, we are in $s-1$. Then we form expectations given all the information we have in $s-1$. As hours worked l_{s-1} and consumption c_{s-1} are control variables, they are known in $s-1$. Hence, using an expectations operator E_{s-1}, we can write

$$\begin{aligned} E_{s-1} \left\{ \frac{\partial}{\partial c_{s-1}} u\left(c_{s-1}, T - l_{s-1}\right) \right\} &= \\ \frac{\partial}{\partial c_{s-1}} u\left(c_{s-1}, T - l_{s-1}\right) &= \beta E_{s-1} \left\{ \left(1 + r_s\right) \frac{\partial}{\partial c_s} u\left(c_s, T - l_s\right) \right\}. \end{aligned}$$

Economically speaking, optimal consumption-saving behaviour requires marginal utility from consumption by each family member in $s-1$ to equal marginal utility in s, corrected for impatience and the interest rate. This condition is similar to the one in (9.1.8), only that here, we have utility from leisure as well.

- Labour-leisure choice

Let us now look at the second first-order condition (9.4.2) to understand the condition for optimal intra-temporal labour supply. When an individual makes the decision of how much to work in s, she actually finds herself in s. Expectations are therefore formed in s and replacing t by s and rearranging slightly, the condition becomes,

$$E_s\left\{\left(\frac{\partial}{\partial\left(T-l_s\right)}u\left(c_s,T-l_s\right)\right)n_s\right\}=E_s\left\{\lambda_s n_s w_s\right\}.$$

As none of the variables in the curly brackets are random from the perspective of s, after removing the expectations operator and cancelling n_s on both sides, we obtain, $\frac{\partial}{\partial(T-l_s)}u\left(c_s,T-l_s\right)=\lambda_s w_s$. We finally use (9.4.4) expressed for $t=s$ and with n_s cancelled to obtain an expression for the shadow price λ_s, $\frac{\partial}{\partial c_s}u\left(c_s,T-l_s\right)=\lambda_s$. Inserting this yields

$$\frac{\frac{\partial}{\partial(T-l_s)}u\left(c_s,T-l_s\right)}{\frac{\partial}{\partial c_s}u\left(c_s,T-l_s\right)}=w_s.$$

This is the standard condition known from static models. It also holds here as the labour-leisure choice is a decision made in each period and has no intertemporal dimension. The trade-off is entirely intra-temporal. In optimum it requires that the ratio of marginal utility of leisure to marginal utility of consumption is given by the ratio of the price of leisure - the wage w_s - to the price of one unit of the consumption good - which is 1 here. Whether an increase in the price of leisure implies an increase in labour supply depends on properties of the instantaneous utility function $u\left(.\right)$. If the income effect caused by higher w_s dominates the substitution effect, a higher wage would imply fewer hours worked. More on this can be found in many introductory textbooks to Microeconomics.

9.5 Solving by substitution

This section is on how to do without dynamic programming. We will get to know a method that allows us to solve stochastic intertemporal problems in discrete time without dynamic programming. Once this chapter is over, you will ask yourself why dynamic programming exists at all ...

9.5.1 Intertemporal utility maximization

The objective is again $\max_{\{c_t\}} E_0\Sigma_{t=0}^{\infty}\beta^t u\left(c_t\right)$ subject to the constraint $a_{t+1}=\left(1+r_t\right)a_t+w_t-c_t$. The household's control variables are $\{c_t\}$, the state variable is a_t, the interest rate r_t and the wage w_t are exogenously given. Now rewrite the objective function and insert the constraint twice,

$$
\begin{aligned}
& E_0 \left\{ \Sigma_{t=0}^{s-1} \beta^t u(c_t) + \beta^s u(c_s) + \beta^{s+1} u(c_{s+1}) + \Sigma_{t=s+2}^{\infty} \beta^t u(c_t) \right\} \\
& = E_0 \Sigma_{t=0}^{s-1} \beta^t u(c_t) + E_0 \beta^s u((1+r_s)a_s + w_s - a_{s+1}) \\
& + E_0 \beta^{s+1} u((1+r_{s+1})\, a_{s+1} + w_{s+1} - a_{s+2}) + E_0 \Sigma_{t=s+2}^{\infty} \beta^t u(c_t) .
\end{aligned}
$$

Note that the expectations operator always refers to knowledge available at the beginning of the planning horizon, i.e. to $t = 0$.

Now compute the first-order condition with respect to a_{s+1}. This is unusual as we directly choose the state variable which is usually understood to be only indirectly influenced by the control variable. Clearly, however, this is just a convenient trick: by choosing a_{s+1}, which is a state variable, we really choose c_s. The derivative with respect to a_{s+1} yields

$$E_0 \beta^s u'(c_s) = E_0 \beta^{s+1} u'(c_{s+1})\,(1 + r_{s+1}).$$

This almost looks like the standard optimal consumption rule. The difference lies in the expectations operator being present on both sides. This is not surprising as we optimally chose a_{s+1} (i.e. c_s), knowing only the state of the system in $t = 0$.

If we now assume we are in s, our expectations would be based on knowledge in s and we could replace E_0 by E_s. We would then obtain $E_s \beta^s u'(c_s) = \beta^s u'(c_s)$ for the left-hand side and our optimality rule reads

$$u'(c_s) = \beta E_s u'(c_{s+1})\,(1 + r_{s+1}).$$

This is the rule we know from Bellman approaches, provided e.g. in (9.1.8).

9.5.2 Capital asset pricing

Let us now ask how an asset that pays an uncertain return in T periods would be priced. Consider an economy with an asset that pays a return r in each period and one long-term asset which can be sold only after T periods at a price p_T which is unknown today. Assuming that investors behave rationally, i.e. they maximize an intertemporal utility function subject to constraints, the price of the long-term asset can be found most easily by using a Lagrange approach or by straightforward inserting.

We assume that an investor maximizes her expected utility $E_0 \Sigma_{t=0}^{T} \beta^t u(c_t)$ subject to the constraints

$$
\begin{aligned}
c_0 + m_0 p_0 + a_0 &= w_0, \\
c_t + a_t &= (1+r) a_{t-1} + w_t, \qquad 1 \leq t \leq T-1, \\
c_T &= m_0 p_T + (1+r)\, a_{T-1} + w_T.
\end{aligned}
$$

In period zero, the individual uses labour income w_0 to pay for consumption goods c_0, to buy m_0 units of the long-term asset and for "normal" assets a_0. In periods one to $T-1$, the individual uses her assets a_{t-1} plus returns r on assets and her wage income w_t to finance consumption and again buy assets a_t (or keep those from the previous period). In the final period T, the long-term asset has a price of p_T and is sold. Wealth from the previous period plus interest plus labour income w_T are further sources of income to pay for consumption c_T. Hence, the individual's control variables are consumption levels c_t and the number m_0 of long-term assets.

This maximization problem can be solved most easily by inserting consumption levels for each period into the objective function. The objective function then reads

$$u(w_0 - m_0 p_0 - a_0) + E_0 \Sigma_{t=1}^{T-1} \beta^t u(w_t + (1+r)a_{t-1} - a_t) \\ + E_0 \beta^T u(m_0 p_T + (1+r)a_{T-1} + w_T).$$

What could now be called control variables are the wealth holdings a_t in periods $t = 0, ..., T-1$ and (as in the original setup) the number of assets m_0 bought in period zero.

Let us now look at the first-order conditions. The first-order condition for wealth in period zero is

$$u'(c_0) = (1+r)\beta E_0 u'(c_1). \tag{9.5.1}$$

Wealth holdings in any period $t > 0$ are optimally chosen according to

$$E_0 \beta^t u'(c_t) = E_0 \beta^{t+1}(1+r)u'(c_{t+1}) \Leftrightarrow E_0 u'(c_t) = \beta(1+r) E_0 u'(c_{t+1}). \tag{9.5.2}$$

We can insert (9.5.2) into the first-period condition (9.5.1) sufficiently often and find

$$u'(c_0) = (1+r)^2 \beta^2 E_0 u'(c_2) = ... = (1+r)^T \beta^T E_0 u'(c_T) \tag{9.5.3}$$

The first-order condition for the number of assets is

$$p_0 u'(c_0) = \beta^T E_0 u'(c_T) p_T. \tag{9.5.4}$$

When we insert combined first-order conditions (9.5.3) for wealth holdings into the first-order condition (9.5.4) for assets, we obtain

$$p_0(1+r)^T \beta^T E_0 u'(c_T) = \beta^T E_0 u'(c_T) p_T \Leftrightarrow \\ p_0 = (1+r)^{-T} E_0 \frac{u'(c_T)}{E_0 u'(c_T)} p_T.$$

which is an equation where we see the analogy to the two period example in ch. 8.3.1 nicely. Instead of $p_{j0} = \frac{1}{1+r} E \frac{u'(c_1)}{Eu'(c_1)} p_{j1}$ in (8.3.1) where we discount by one period only and evaluate returns at expected marginal utility in period one, we discount by T periods and evaluate returns at marginal utility in period T.

This equation also offers a lesson for life when we assume risk-neutrality for simplicity: if the payoff p_T from a long-term asset is not high enough such that the current price is higher than the present value of the payoff, $p_0 > (1+r)^{-T} E_0 p_T$, then the long-term asset is simply dominated by short-term investments that pay a return of r per period. Optimal behaviour would imply not buying the long-term asset and just putting wealth into "normal" assets. This should be kept in mind the next time you talk to your insurance agent who tries to sell you life-insurance or private pension plans. Just ask for the present value of the payoffs and compare them to the present value of what you pay into the savings plan.

9.5.3 Sticky prices

- The setup

Sticky prices are a fact of life. In macroeconomic models, they are either assumed right away, or assumed following Calvo price setting or result from some adjustment-cost setup. Here is a simplified way to derive sluggish price adjustment based on adjustment cost.

The firm's objective function is to maximize its present value defined by the sum of discounted expected profits,

$$V_t = E_t \Sigma_{\tau=t}^{\infty} \left(\frac{1}{1+r} \right)^{\tau-t} \pi_\tau .$$

Profits at a point $\tau \geq t$ in time are given by

$$\pi_\tau = p_\tau x_\tau - w_\tau l_\tau - \Phi(p_\tau, p_{\tau-1})$$

where $\Phi(p_\tau, p_{\tau-1})$ are price adjustment costs. These are similar in spirit to the adjustment costs presented in ch. 5.5.1. We will later use a specification given by

$$\Phi(p_\tau, p_{\tau-1}) = \frac{\phi}{2} \left[\frac{p_\tau - p_{\tau-1}}{p_{\tau-1}} \right]^2 . \tag{9.5.5}$$

This specification captures the essential mechanism that is required to make prices sticky, there are increasing costs in the difference $p_\tau - p_{\tau-1}$. The fact that the price change is squared is not essential - in fact, as with all adjustment cost mechanisms, any power larger than 1 would do the job. More care about economic implications needs to be taken when a reasonable model is to be specified.

The firm uses a technology

$$x_\tau = A_\tau l_\tau .$$

We assume that there is a certain demand elasticity ε for the firm's output. This can reflect a monopolistic competition setup. The firm can choose its output x_τ at each point in time freely by hiring the corresponding amount of labour l_τ. Labour productivity A_τ or other quantities can be uncertain.

- Solving by substitution

Inserting everything into the objective function yields

$$\begin{aligned} V_t &= E_t \Sigma_{\tau=t}^{\infty} \left(\frac{1}{1+r} \right)^{\tau-t} \left(p_\tau x_\tau - \frac{w_\tau}{A_\tau} x_\tau - \Phi(p_\tau, p_{\tau-1}) \right) \\ &= E_t \left(p_t x_t - \frac{w_t}{A_t} x_t - \Phi(p_t, p_{t-1}) \right) + E_t \left(p_{t+1} x_{t+1} - \frac{w_{t+1}}{A_{t+1}} x_{t+1} - \Phi(p_{t+1}, p_t) \right) \\ &+ E_t \Sigma_{\tau=t+2}^{\infty} \left(\frac{1}{1+r} \right)^{\tau-t} \left(p_\tau x_\tau - \frac{w_\tau}{A_\tau} x_\tau - \Phi(p_\tau, p_{\tau-1}) \right) . \end{aligned}$$

The second and third line present a rewritten method which allows us to see the intertemporal structure of the maximization problem better. We now maximize this objective function by choosing output x_t for today. (Output levels in the future are chosen at a later stage.)

The first-order condition is

$$E_t\left(\frac{d\left[p_t x_t\right]}{dx_t}-\frac{w_t}{A_t}-\frac{d\Phi\left(p_t,p_{t-1}\right)}{dx_t}\right)-E_t\frac{d\Phi\left(p_{t+1},p_t\right)}{dx_t}=0\Leftrightarrow$$
$$\frac{d\left[p_t x_t\right]}{dx_t}=\frac{w_t}{A_t}+\frac{d\Phi\left(p_t,p_{t-1}\right)}{dx_t}+E_t\frac{d\Phi\left(p_{t+1},p_t\right)}{dx_t}.$$

It has certain well-known components and some new ones. If there were no adjustment costs, i.e. $\Phi(.)=0$, the intertemporal problem would become a static one and the usual condition would equate marginal revenue $d\left[p_t x_t\right]/dx_t$ with marginal cost w_t/A_t. With adjustment costs, however, a change in output today not only affects adjustment costs today $\frac{d\Phi(p_t,p_{t-1})}{dx_t}$ but also (expected) adjustment costs $E_t\frac{d\Phi(p_{t+1},p_t)}{dx_t}$ tomorrow. As all variables with index t are assumed to be known in t, expectations are formed only with respect to adjustment costs tomorrow in $t+1$.

Specifying the adjustment cost function as in (9.5.5) and computing marginal revenue using the demand elasticity ε_t gives

$$\frac{d\left[p_t x_t\right]}{dx_t}=\frac{w_t}{A_t}+\frac{d}{dx_t}\left[\frac{\phi}{2}\left[\frac{p_t-p_{t-1}}{p_{t-1}}\right]^2\right]+E_t\frac{d}{dx_t}\left[\frac{\phi}{2}\left[\frac{p_{t+1}-p_t}{p_t}\right]^2\right]\Leftrightarrow$$
$$\frac{dp_t}{dx_t}x_t+p_t=\left(1+\varepsilon_t^{-1}\right)p_t=\frac{w_t}{A_t}+\phi\left[\frac{p_t-p_{t-1}}{p_{t-1}}\right]\frac{1}{p_{t-1}}\frac{dp_t}{dx_t}+E_t\left\{\phi\left[\frac{p_{t+1}-p_t}{p_t}\right]\frac{d}{dx_t}\frac{p_{t+1}}{p_t}\right\}$$
$$=\frac{w_t}{A_t}+\phi\left[\frac{p_t-p_{t-1}}{p_{t-1}}\right]\frac{p_t}{x_t p_{t-1}}\varepsilon_t^{-1}-E_t\left\{\phi\left[\frac{p_{t+1}-p_t}{p_t}\right]\frac{p_{t+1}}{p_t^2}\frac{dp_t}{dx_t}\right\}\Leftrightarrow$$
$$\left(1+\varepsilon_t^{-1}\right)p_t=\frac{w_t}{A_t}+\phi\left[\frac{p_t-p_{t-1}}{p_{t-1}}\right]\frac{p_t}{x_t p_{t-1}}\varepsilon_t^{-1}-E_t\left\{\phi\left[\frac{p_{t+1}-p_t}{p_t}\right]\frac{p_{t+1}}{p_t^2}\frac{p_t}{x_t}\varepsilon_t^{-1}\right\}\Leftrightarrow$$
$$\left(1+\varepsilon_t^{-1}\right)p_t=\frac{w_t}{A_t}+\phi\left[\frac{p_t-p_{t-1}}{p_{t-1}}\right]\frac{p_t}{x_t p_{t-1}}\varepsilon_t^{-1}-\frac{1}{p_t x_t}\varepsilon_t^{-1}\phi E_t\left\{\frac{p_{t+1}-p_t}{p_t}p_{t+1}\right\}$$

where $\varepsilon_t\equiv\frac{dx_t}{dp_t}\frac{p_t}{x_t}$ is the demand elasticity for the firm's good. Again, $\phi=0$ would give the standard static optimality condition $\left(1+\varepsilon_t^{-1}\right)p_t=w_t/A_t$ where the price is a markup over marginal cost. With adjustment costs, prices change only slowly.

9.5.4 Optimal employment with adjustment costs

- The setup

Consider a firm that maximizes profits as in ch. 5.5.1,

$$\Pi_t=E_t\Sigma_{\tau=t}^{\infty}\frac{1}{(1+r)^{\tau-t}}\pi_\tau. \tag{9.5.6}$$

We are today in t, time extends until infinity and the time between today and infinity is denoted by τ. There is no particular reason why the planning horizon is infinity in contrast to ch. 5.5.1. Here we will stress that the optimality condition for employment is identical to a finite horizon problem.

Instantaneous profits are given by the difference between revenue in t, which is identical to output $F(L_t)$ with an output price normalized to unity, labour cost $w_t L_t$ and adjustment cost $\Phi(L_t - L_{t-1})$,

$$\pi_t = F(L_t) - w_t L_t - \Phi(L_t - L_{t-1}). \tag{9.5.7}$$

Costs induced by the adjustment of the number of employees between the previous period and today are captured by $\Phi(.)$. Usually, one assumes costs both from hiring and from firing individuals, i.e. both for an increase in the labour force, $L_t - L_{t-1} > 0$, and from a decrease, $L_t - L_{t-1} < 0$. A simple functional form for $\Phi(.)$ which captures this idea is a quadratic form, i.e. $\Phi(L_t - L_{t-1}) = \frac{\phi}{2}(L_t - L_{t-1})^2$, where ϕ is a constant.

Uncertainty for a firm can come from many sources: Uncertain demand, uncertainty concerning the production process, uncertainty over labour costs or other sources. As we express profits in units of the output good, we assume that the real wage w_t, i.e. the amount of output goods to be paid to labour, is uncertain. Adjustment cost $\Phi(L_t - L_{t-1})$ are certain, i.e. the firm knows how many units of output profits reduce by when employment changes by $L_t - L_{t-1}$.

As in static models of the firm, the control variable of the firm is employment L_t. In contrast to static models, however, employment decisions today in t not only affects employment today but also employment tomorrow as the employment decision in t affects adjustment costs in $t+1$. There is therefore an intertemporal link the firm needs to take into account which is not present in the firm's static models.

- Solving by substitution

This maximization problem can be solved directly by inserting (9.5.7) into the objective function (9.5.6). One can then choose optimal employment for some point in time $t \leq s < \infty$ after having split the objective function into several subperiods - as for example in the previous chapter 9.5.1. The solution reads (to be shown in exercise 7)

$$F'(L_t) = w_t + \Phi'(L_t - L_{t-1}) - E_t \frac{\Phi'(L_{t+1} - L_t)}{1+r}.$$

When employment L_t is chosen in t, there is only uncertainty concerning L_{t+1}. The current wage w_t (and all other deterministic quantities as well) are known with certainty. L_{t+1} is uncertain, however, from the perspective of today as the wage in $\tau + 1$ is unknown and $L_{\tau+1}$ will be have to be adjusted accordingly in $\tau + 1$. Hence, expectations apply only to the adjustment-cost term which refers to adjustment costs which occur in period $t+1$. Economically speaking, given employment L_{t-1} in the previous period, employment in t is chosen such that marginal productivity of labour equals labour costs adjusted for current and expected future adjustment costs. Expected future adjustment costs are discounted by the interest rate r to obtain its present value.

When we specify the adjustment cost function as a quadratic function, $\Phi(L_t - L_{t-1}) = \frac{\phi}{2}[L_t - L_{t-1}]^2$, we obtain

$$F'(L_t) = w_t + \phi[L_t - L_{t-1}] - E_t \frac{\phi[L_{t+1} - L_t]}{1+r}.$$

If there were no adjustment costs, i.e. $\phi = 0$, we would have $F'(L_t) = w_t$. Employment would be chosen such that marginal productivity equals the real wage. This confirms the initial statement that the intertemporal problem of the firm arises purely from the adjustment costs. Without adjustment costs, i.e. with $\phi = 0$, the firm has the "standard" instantaneous, period-specific optimality condition.

9.6 An explicit time path for a boundary condition

Sometimes, an explicit time path for optimal behaviour is required. The transversality condition is then usually not very useful. A more pragmatic approach sets assets at some future point in time at some exogenous level. This allows us to then (at least numerically) compute the optimal path for all points in time before this final point easily.

Let T be the final period of life in our model, i.e. set $a_{T+1} = 0$ (or some other level for example the deterministic steady state level). Then, from the budget constraint, we can deduce consumption in T,

$$a_{T+1} = (1 + r_T) a_T + w_T - c_T \Leftrightarrow c_T = (1 + r_T) a_T + w_T.$$

Optimal consumption in $T-1$ still needs to obey the Euler equation, compare for example to (9.1.9), i.e.

$$u'(c_{T-1}) = E_{T-1}\beta[1 + r_T] u'(c_T).$$

As the budget constraint requires

$$a_T = (1 + r_{T-1})\ a_{T-1} + w_{T-1} - c_{T-1},$$

optimal consumption in $T-1$ is determined by

$$u'(c_{T-1}) = E_{T-1}\beta[1 + r_T] u'((1 + r_T)[(1 + r_{T-1}) a_{T-1} + w_{T-1} - c_{T-1}] + w_T)$$

This is one equation in one unknown, c_{T-1}, where expectations need to be formed about r_T and w_T and w_{T-1} are unknown. When we assume a probability distribution for r_T and w_T, we can replace E_{T-1} by a summation over states and solve this expression numerically in a straightforward way.

9.7 Further reading and exercises

A recent introduction and detailed analysis of discrete time models with uncertainty in the real business cycle tradition with homogeneous and heterogenous agents is by Heer and Mausner (2005). Stokey and Lucas take a more rigorous approach to the one taken here (1989). An almost comprehensive in-depth presentation of macroeconomic aspects under uncertainty is provided by Ljungqvist and Sargent (2004).

On capital asset pricing in one-sector economies, references include Jermann (1998), Danthine, Donaldson and Mehra (1992), Abel (1990), Rouwenhorst (1995), Stokey and Lucas (1989, ch. 16.2) and Lucas (1978). An overview is in ch. 13 of Ljungqvist and Sargent (2004).

The example for sticky prices is inspired by Ireland (2004), going back to Rotemberg (1982).

The statement that "the predictions concerning real variables do not change when a numéraire good is not chosen" is not as obvious as it might appear from remembering Walras' law from undergraduate micro courses. There is a literature that analyses the effects of the choice of numéraire for real outcomes for the economy when there is imperfect competition. See e.g. Gabszewicz and Vial (1972) or Dierker and Grodahl (1995).

Exercises Chapter 9
Applied Intertemporal Optimization
Discrete time infinite horizon models under uncertainty

1. Central planner
   Consider an economy where output is produced by
   $$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$
   Again, as in the OLG example in equation (8.1.1), total factor productivity A_t is stochastic. Let capital evolve according to
   $$K_{t+1} = (1-\delta) K_t + Y_t - C_t$$
   The central planner maximizes
   $$\max_{\{C_\tau\}} E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau)$$
   by again choosing a path of aggregate consumption flows C_τ. At t, all variables indexed t are known. The only uncertainty concerns A_{t+1}. What are the optimality conditions?

2. A household maximization problem
   Consider the optimal saving problem of the household in ch. 9.3. Derive the Euler equation (9.3.6).

3. Endogenous labour supply
   Solve the endogenous labour supply setup in ch. 9.4 by using dynamic programming.

4. Closed-form solution
   Solve this model for the utility function $u(C) = \frac{C^{1-\sigma}-1}{1-\sigma}$ and for $\delta = 1$. Solve it for a more general case (Benhabib and Rustichini, 1994).

5. Habit formation
   Assume instantaneous utility depends, not only on current consumption, but also on habits (see for example Abel, 1990). Let the utility function therefore look like
   $$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, v_\tau),$$

where v_τ stands for habits like e.g. past consumption, $v_\tau = v(c_{\tau-1}, c_{\tau-2}, ...)$. Let such an individual maximize utility subject to the budget constraint

$$a_{t+1} = (1 + r_t) a_t + w_t - p_t c_t$$

(a) Assume the individual lives in a deterministic world and derive a rule for an optimal consumption path where the effect of habits are explicitly taken into account. Specify habits by $v_\tau = c_{\tau-1}$.

(b) Let there be uncertainty with respect to future prices. At a point in time t, all variables indexed by t are known. What is the optimal consumption rule when habits are treated in a parametric way?

(c) Choose a plausible instantaneous utility function and discuss the implications for optimal consumption given habits $v_\tau = c_{\tau-1}$.

6. Risk-neutral valuation
Under which conditions is there a risk neutral valuation relationship for contingent claims in models with many periods?

7. Labour demand under adjustment cost
Solve the maximization problem of the firm in ch. 9.5.4 by directly inserting profits (9.5.7) into the objective function (9.5.6) and then choosing L_t.

8. Solving by substitution
Solve the problem from ch. 9.5 in a slightly extended version, i.e. with prices p_t. Maximize $E_0 \Sigma_{t=0}^{\infty} \beta^t u(c_t)$ by choosing a time path $\{c_t\}$ for consumption subject to $a_{t+1} = (1 + r_t) a_t + w_t - p_t c_t$.

9. Matching on labour markets
Let employment L_t in a firm follow

$$L_{t+1} = (1 - s) L_t + \mu V_t,$$

where s is a constant separation rate, μ is a constant matching rate and V_t denotes the number of jobs a firm currently offers. The firm's profits π_τ in period τ are given by the difference between revenue $p_\tau Y(L_\tau)$ and costs, where costs stem from wage payments and costs for vacancies V_τ captured by a parameter γ,

$$\pi_\tau = p_\tau Y(L_\tau) - w_\tau L_\tau - \gamma V_\tau.$$

The firm's objective function is given by

$$\Pi_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} \pi_\tau,$$

where β is a discount factor and E_t is the expectations operator.

(a) Assume a deterministic world. Let the firm choose the number of vacancies optimally. Use a Lagrangian to derive the optimality condition. Assume that there is an interior solution. Why is this an assumption that might not always be satisfied from the perspective of a single firm?

(b) Let us now assume that there is uncertain demand which translates into uncertain prices p_τ which are exogenous to the firm. Solve the optimal choice of the firm by inserting all equations into the objective function. Maximize by choosing the state variable and explain also in words what you do. Give an interpretation of the optimality condition. What does it imply for the optimal choice of V_τ?

10. Optimal training for a marathon
Imagine you want to participate in a marathon or any other sports event. It will take place in m days, i.e. in $t+m$ where t is today. You know that taking part in this event requires training e_τ, $\tau \in [t, t+m]$. Unfortunately, you dislike training, i.e. your instantaneous utility $u(e_\tau)$ decreases in effort, $u'(e_\tau) < 0$. On the other hand, training allows you to be successful in the marathon: more effort increases your personal fitness F_τ. Assume that fitness follows $F_{\tau+1} = (1-\delta) F_\tau + e_\tau$, with $0 < \delta < 1$, and fitness at $t+m$ is good for you yielding happiness of $h(F_{t+m})$.

(a) Formally formulate an objective function which captures the trade-offs in such a training program.

(b) Assume that everything is deterministic. How would your training schedule look (the optimal path of e_τ)?

(c) In the real world, normal night life reduces fitness in a random way, i.e. δ is stochastic. How does your training schedule look now?

Part IV

Stochastic models in continuous time

Part IV is the final part of this book and, logically, analyzes continuous time models under uncertainty. The choice between working in discrete or continuous time is partly driven by previous choices: If the literature is mainly in discrete time, students will find it helpful to work in discrete time as well. The use of discrete time methods seem to hold for macroeconomics, at least when it comes to the analysis of business cycles. On the other hand, when we talk about economic growth, labour market analyses and finance, continuous time methods are very prominent.

Whatever the tradition in the literature, continuous time models have the huge advantage that they are analytically generally more tractable, once some initial investment into new methods has been digested. As an example, some papers in the literature have shown that continuous time models with uncertainty can be analyzed in simple phase diagrams as in deterministic continuous time setups. See ch. 10.6 and ch. 11.6 on further reading for references from many fields.

To facilitate access to the magical world of continuous time uncertainty, part IV presents the tools required to work with uncertainty in continuous time models. It is probably the most innovative part of this book as many results from recent research flow directly into it. This part also most strongly incorporates the central philosophy behind writing this book: There will be hardly any discussion of formal mathematical aspects like probability spaces, measurability and the like. While some will argue that one can not work with continuous time uncertainty without having studied mathematics, this chapter and the many applications in the literature prove the opposite. The objective here is to clearly make the tools for continuous time uncertainty available in a language that is accessible for anyone with an interest in these tools and some "feeling" for dynamic models and random variables. The chapters on further reading will provide links to the more mathematical literature. Maybe this is also a good point for the author of this book to thank all the mathematicians who helped him gain access to this magical world. I hope they will forgive me for "betraying their secrets" to those who, maybe in their view, were not appropriately initiated.

Chapter 10 provides the background for optimization problems. As in part II where we first looked at differential equations before working with Hamiltonians, here we will first look at stochastic differential equations. After some basics, the most interesting aspect of working with uncertainty in continuous time follows: Ito's lemma and, more generally, change-of-variable formulas for computing differentials will be presented. As an application of Ito's lemma, we will get to know one of the most famous results in Economics - the Black-Scholes formula. This chapter also presents methods for how to solve stochastic differential equations or how to verify solutions and compute moments of random variables described by a stochastic process.

Chapter 11 then looks once more at maximization problems. We will get to know the classic intertemporal utility maximization problem both for Poisson uncertainty and for Brownian motion. The chapter also shows the link between Poisson processes and matching models of the labour market. This is very useful for working with extensions of the simple matching model that allows for savings. Capital asset pricing and natural volatility conclude the chapter.

Chapter 10

SDEs, differentials and moments

When working in continuous time, uncertainty enters the economy usually in the form of Brownian motion, Poisson processes or Levy processes. This uncertainty is represented in economic models by stochastic differential equations (SDEs) which describe for example the evolution of prices or technology frontiers. This section will cover a wide range of differential equations (and show how to work with them) that appear in economics and finance. It will also show how to work with functions of stochastic variables, for example how output evolves given that TFP is stochastic or how wealth of a household grows over time, given that the price of the asset held by the household is random. The entire treatment here, as before in this book, will be non-rigorous and will focus on "how to compute things".

10.1 Stochastic differential equations (SDEs)

10.1.1 Stochastic processes

We got to know random variables in ch. 7.1. A random variable relates in some loose sense to a stochastic process of how (deterministic) static models relate to (deterministic) dynamic models: Static models describe one equilibrium, dynamic models describe a sequence of equilibria. A random variable has, "when looked at once" (e.g. when throwing a die once), one realization. A stochastic process describes a sequence of random variables and therefore, "when looked at once", describes a sequence of realizations. More formally, we have the following:

Definition 10.1.1 *(Ross, 1996) A stochastic process is a parameterized collection of random variables*

$$\{X(t)\}_{t\in[t_0,T]}.$$

Let us look at an example for a stochastic process. We start from the normal distribution of ch. 7.2.2 whose mean and variance are given by μ and σ^2 and its density function is

$f(z) = \left(\sqrt{2\pi\sigma^2}\right)^{-1} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2}$. Now define a normally distributed random variable $Z(t)$ that has a variance that is a function of some t: instead of σ^2, write $\sigma^2 t$. Hence, the random variables we just defined have as density function $f(z) = \left(\sqrt{2\pi\sigma^2 t}\right)^{-1} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma\sqrt{t}}\right)^2}$. By having done so and by interpreting t as time, $Z(t)$ is in fact a stochastic process: we have a collection of random variables, all normally distributed, they are parameterized by time t.

Stochastic processes can be stationary, weakly stationary or non-stationary. Stationarity is a more restrictive concept than weak stationarity.

Definition 10.1.2 *(Ross, 1996, ch. 8.8): A process $X(t)$ is stationary if $X(t_1), ...,$ $X(t_n)$ and $X(t_1+s), ..., X(t_n+s)$ have the same joint distribution for all n and s.*

An implication of this definition, which might help to get some "feeling" for this definition, is that a stationary process $X(t)$ implies that, being in $t = 0$, $X(t_1)$ and $X(t_2)$ have the same distribution for all $t_2 > t_1 > 0$. A weaker concept of stationarity only requires the first two moments of $X(t_1)$ and $X(t_2)$ (and a condition on the covariance) to be satisfied.

Definition 10.1.3 *(Ross, 1996) A process $X(t)$ is weakly stationary if the first two moments are the same for all t and the covariance between $X(t_2)$ and $X(t_1)$ depends only on $t_2 - t_1$,*

$$E_0 X(t) = \mu, \quad Var X(t) = \sigma^2, \quad Cov(X(t_2), X(t_1)) = f(t_2 - t_1),$$

where μ and σ^2 are constants and $f(.)$ is some function.

Definition 10.1.4 *A process which is neither stationary nor weakly stationary is non-stationary.*

Probably the best-known stochastic process in continuous time is the Brownian motion. It is sometimes called the Wiener process after the mathematician Wiener who provided the following definition.

Definition 10.1.5 *(Ross, 1996) Brownian motion*
A stochastic process $z(t)$ is a Brownian motion process if (i) $z(0) = 0$, (ii) the process has stationary independent increments and (iii) for every $t > 0$, $z(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

The first condition $z(0) = 0$ is a normalization. Any $z(t)$ that starts at, say z_0, can be redefined as $z(t) - z_0$. The second condition says that for $t_4 > t_3 \geq t_2 > t_1$ the increment $z(t_4) - z(t_3)$, which is a random variable, is independent of previous increments, say $z(t_2) - z(t_1)$. 'Independent increments' implies that Brownian motion is a Markov process. Assuming that we are in t_3 today, the distribution of $z(t_4)$ depends only on

$z(t_3)$, i.e. on the current state, and not on previous states like $z(t_1)$. Increments are said to be stationary if, according to the above definition of stationarity, the stochastic process $X(t) \equiv z(t) - z(t-s)$ where s is a constant, has the same distribution for any t. Finally, the third condition is the heart of the definition - $z(t)$ is normally distributed. The variance increases linearly in time; the Wiener process is therefore non-stationary.

Let us now define a stochastic process which plays also a major role in economics.

Definition 10.1.6 *Poisson process (adapted following Ross 1993, p. 210)*
A stochastic process $q(t)$ is a Poisson process with arrival rate λ if (i) $q(0) = 0$, (ii) the process has independent increments and (iii) the increment $q(\tau) - q(t)$ in any interval of length $\tau - t$ (the number of "jumps") is Poisson distributed with mean $\lambda[\tau - t]$, i.e. $q(\tau) - q(t) \sim Poisson(\lambda[\tau - t])$.

A Poisson process (and other related processes) are also sometimes called "counting processes" as $q(t)$ counts how often a jump has occurred, i.e. how often something has happened.

There is a close similarity in the first two points of this definition with the definition of Brownian motion. The third point here means more precisely that the probability that the process increases n times between t and $\tau > t$ is given by

$$P[q(\tau) - q(t) = n] = e^{-\lambda[\tau - t]} \frac{(\lambda[\tau - t])^n}{n!}, \qquad n = 0, 1, ... \tag{10.1.1}$$

We know this probability from the definition of the Poisson distribution in ch. 7.2.1. This is probably where the Poisson process got its name from. Hence, one could think of as many stochastic processes as there are distributions, defining each process by the distribution of its increments.

The most common way to present Poisson processes is by looking at the distribution of the increment $q(\tau) - q(t)$ over a very small time interval $[t, \tau]$. The increment $q(\tau) - q(t)$ for τ very close to t is usually expressed by $dq(t)$. A stochastic process $q(t)$ is then a Poisson process if its increment $dq(t)$ is driven by

$$dq(t) = \begin{cases} 0 \text{ with probability } 1-\lambda dt \\ 1 \text{ with probability } \lambda dt \end{cases}, \tag{10.1.2}$$

where the parameter λ is again called the arrival rate. A high λ then means that the process jumps on average more often than with a low λ.

While this presentation is intuitive and widely used, one should note that the probabilities given in (10.1.2) are an approximation of the ones in (10.1.1) for $\tau - t = dt$, i.e. for very small time intervals. We will return to this below in ch. 10.5.2, see Poisson process II.

These stochastic processes (and other processes) can now be combined in various ways to construct more complex processes. These more complex processes can be well represented by stochastic differential equations (SDEs).

10.1.2 Stochastic differential equations

The most frequently used SDEs include Brownian motion as the source of uncertainty. These SDEs are used to model for example the evolution of asset prices or budget constraints of households. Other examples include SDEs with Poisson uncertainty used explicitly in the natural volatility literature, in finance, labour markets, international macro or in other contexts mentioned above. Finally and more recently, Levy processes are used in finance as they allow for a much wider choice of properties of distributions of asset returns than, let us say, Brownian motion. We will now get to know examples for each type.

For all Brownian motions that will follow, we will assume, unless explicitly stated otherwise, that increments have a standard normal distribution, i.e. $E_t [z(\tau) - z(t)] = 0$ and $\text{var}_t [z(\tau) - z(t)] = \tau - t$. We will call this standard Brownian motion. It is therefore sufficient, consistent with most papers in the literature and many mathematical textbooks, to work with a normalization of σ in definition 10.1.5 of Brownian motion to 1.

- Brownian motion with drift

This is one of the simplest SDEs. It reads

$$dx(t) = adt + bdz(t). \tag{10.1.3}$$

The constant a can be called drift rate, b^2 is sometimes referred to as the variance rate of $x(t)$. In fact, ch. 10.5.4 shows that the expected increase of $x(t)$ is determined by a only (and not by b). In contrast, the variance of $x(\tau)$ for some future $\tau > t$ is only determined by b. The drift rate a is multiplied by dt, a "short" time interval, the variance parameter b is multiplied by $dz(t)$, the increment of the Brownian motion process $z(t)$ over a small time interval. This SDE (and all the others following later) therefore consist of a deterministic part (the dt-term) and a stochastic part (the dz-term).

An intuition for this differential equation can be most easily gained by undertaking a comparison with a deterministic differential equation. If we neglected the Wiener process for a moment (set $b = 0$), divide by dt and rename the variable as y, we obtain the simple ordinary differential equation

$$\dot{y}(t) = a \tag{10.1.4}$$

whose solution is $y(t) = y_0 + at$. When we draw this solution and also the above SDE for three different realizations of $z(t)$, we obtain the following figure.

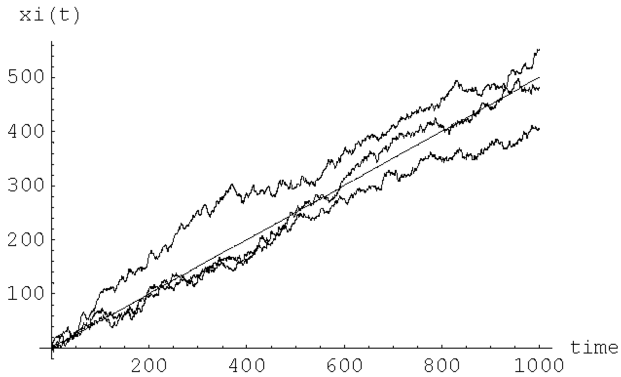


Figure 10.1.1 *The solution of the deterministic differential equation (10.1.4) and three realizations of the related stochastic differential equation (10.1.3)*

Hence, intuitively speaking, adding a stochastic component to the differential equation leads to fluctuations around the deterministic path. Clearly, how much the solution of the SDE differs from the deterministic one is random, i.e. unknown. Further below in ch. 10.5.4, we will understand that the solution of the deterministic differential equation (10.1.4) is identical to the evolution of the expected value of $x(t)$, i.e. $y(t) = E_0 x(t)$ for $t > 0$.

- Generalized Brownian motion (Ito processes)

A more general way to describe stochastic processes is the following SDE

$$dx(t) = a(x(t), z(t), t)\, dt + b(x(t), z(t), t)\, dz(t). \tag{10.1.5}$$

Here, one also refers to $a(.)$ as the drift rate and to $b^2(.)$ as the instantaneous variance rate. Note that these functions can be stochastic themselves. In addition to arguments $x(t)$ and time, Brownian motion $z(t)$ can be included in these arguments. Thinking of (10.1.5) as a budget constraint of a household, an example could be that wage income or the interest rate depend on the current realization of the economy's fundamental source of uncertainty, which is $z(t)$.

- Stochastic differential equations with Poisson processes

Differential equations that are driven by a Poisson process can, of course, also be constructed. A very simple example is

$$dx(t) = adt + bdq(t). \tag{10.1.6}$$

A realization of this path for $x(0) = x_0$ is depicted in the following figure and can be understood very easily. As long as no jump occurs, i.e. as long as $dq = 0$, the variable $x(t)$ follows $dx(t) = adt$ which means linear growth, $x(t) = x_0 + at$. This is plotted as the thin line. When q jumps, i.e. $dq = 1$, $x(t)$ increases by b : writing $dx(t) = \tilde{x}(t) - x(t)$, where $\tilde{x}(t)$ is the level of x immediately after the jump, and letting the jump be "very fast" such that $dt = 0$ during the jump, we have $\tilde{x}(t) - x(t) = b \cdot 1$, where the 1 stems from $dq(t) = 1$. Hence,

$$\tilde{x}(t) = x(t) + b. \tag{10.1.7}$$

Clearly, the points in time when a jump occurs are random. A tilde (˜) will always denote in what (and in various papers in the literature) follows the value of a quantity immediately after a jump.

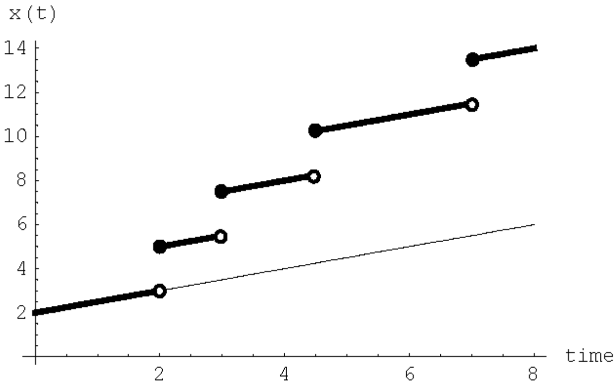


Figure 10.1.2 *An example of a Poisson process with drift (thick line) and a deterministic differential equation (thin line)*

In contrast to Brownian motion, a Poisson process contributes to the increase of the variable of interest: without the $dq(t)$ term (i.e. for $b = 0$), $x(t)$ would follow the thin line. With occasional jumps, $x(t)$ grows faster. In the Brownian motion case of the figure before, realizations of $x(t)$ remained "close to" the deterministic solution. This is simply due to the fact that the expected increment of Brownian motion is zero while the expected increment of a Poisson process is positive.

Note that in the more formal literature, the tilde is not used but a difference is made between $x(t)$ and $x(t_-)$ where t_- stands for the point in time an "instant" before t. (This is probably easy to understand on an intuitive level, thinking about it for too long might not be a good idea as time is continuous ...) The process $x(t)$ is a so called *càdlàg* process. The expression càdlàg is an acronym from the french "continue à droite, limites à gauche". That is, the paths of $x(t)$ are continuous from the right with left limits. This is captured in the above figure by the black dots (continuous from the right) and the white circles (limits from the left). With this notation, one would express the change in x due to a jump by $x(t) = x(t_-) + b$ as the value of x to which b is added is the value of x before the jump. As the tilde-notation turned out to be relatively intuitive, we will follow it in what follows.

- A geometric Poisson process

An further example would be the geometric Poisson process

$$dx(t) = a(q(t), t)\, x(t)\, dt + b(q(t), t)\, x(t)\, dq(t). \tag{10.1.8}$$

Processes are usually called geometric when they describe the rate of change of some RV $x(t)$, i.e. $dx(t)/x(t)$ is not a function of $x(t)$. In this example, the deterministic part shows that $x(t)$ grows at the rate of $a(.)$ in a deterministic way and jumps by $b(.)$ percent, when $q(t)$ jumps. Note that in contrast to a Brownian motion SDE, $a(.)$ here is *not* the average growth rate of $x(t)$ (see below on expectations).

Geometric Poisson processes as here are sometimes used to describe the evolution of asset prices in a simple way. There is some deterministic growth component $a(.)$ and some stochastic component $b(.)$. When the latter is positive, this could reflect new technologies in the economy. When the latter is negative, this equation could be used to model negative shocks like oil-price shocks or natural disasters.

- Aggregate uncertainty and random jumps

An interesting extension of a Poisson differential equation consists in making the amplitude of the jump random. Taking a simple differential equation with Poisson uncertainty as starting point, $dA(t) = bA(t)\, dq(t)$, where b is a constant, we can now assume that $b(t)$ is governed by some distribution, i.e.

$$dA(t) = b(t)\, A(t)\, dq(t), \qquad \text{where } b(t) \sim \left(\mu, \sigma^2\right). \tag{10.1.9}$$

Assume that $A(t)$ is total factor productivity in an economy. Then, $A(t)$ does not change as long as $dq(t) = 0$. When $q(t)$ jumps, $A(t)$ changes by $b(t)$, i.e. $dA(t) \equiv \tilde{A}(t) - A(t) = b(t)\, A(t)$, which we can rewrite as

$$\tilde{A}(t) = (1 + b(t))\, A(t), \quad \forall t \text{ where } q(t) \text{ jumps.}$$

This equation says that whenever a jump occurs, $A(t)$ increases by $b(t)$ percent, i.e. by the realization of the random variable $b(t)$. Obviously, the realization of $b(t)$ matters only for points in time where $q(t)$ jumps.

Note that (10.1.9) is the stochastic differential equation representation of the evolution of the states of the economy in the Pissarides-type matching model of Shimer (2005), where aggregate uncertainty, here $A(t)$ follows from a Poisson process. The presentation in Shimer's paper is, "A shock hits the economy according to a Poisson process with arrival rate λ, at which point a new pair (p', s') is drawn from a state dependent distribution." (p. 34). Note also that using (10.1.9) and assuming large families such that there is no uncertainty from labour income left on the household level would allow to analyze the effects of saving and thereby capital accumulation over the business cycle in a closed-economy model with risk-averse households. The background for the saving decision would be ch. 11.1.

10.1.3 The integral representation of stochastic differential equations

Stochastic differential equations as presented here can also be represented by integral versions. This is identical to the integral representations for deterministic differential equations in ch. 4.3.3. The integral representation will be used frequently when computing moments of $x(\tau)$. As an example, think of the expected value of x for some future point in time τ, when expectations are formed today in t, i.e. information until t is available, $E_t x(\tau)$. See ch. 10.5.4 or 11.1.6.

- Brownian motion

Consider a differential equation as (10.1.5). It can more rigorously be represented by its integral version,

$$x(\tau) - x(t) = \int_t^\tau a(x,s)ds + \int_t^\tau b(x,s)dz(s). \tag{10.1.10}$$

This version is obtain by first rewriting (10.1.5) as $dx(s) = a(x,s)\,ds + b(x,s)\,dz(s)$, i.e. by simply changing the time index from t to s (and dropping $z(s)$ and writing x instead of $x(s)$ to shorten notation). Applying then the integral $\int_t^\tau$ on both sides gives (10.1.10).

This implies, inter alia, a "differentiation rule"

$$\begin{aligned} d\left[\int_t^\tau a(x,s)ds + \int_t^\tau b(x,s)dz(s)\right] &= d[x(\tau) - x(t)] = dx(\tau) \\ &= a(x,\tau)d\tau + b(x,\tau)dz(\tau). \end{aligned}$$

- Poisson processes

Now consider a generalized version of the SDE in (10.1.6), with again replacing t by s, $dx(s) = a(x(s), q(s))\,ds + b(x(s), q(s))\,dq(s)$. The integral representation reads, after applying $\int_t^\tau$ to both sides,

$$x(\tau) - x(t) = \int_t^\tau a(x(s), q(s))\,ds + \int_t^\tau b(x(s), q(s))\,dq(s).$$

This can be checked by computing the differential with respect to time τ.

10.2 Differentials of stochastic processes

Possibly the most important aspect when working with stochastic processes in continuous time is that rules for computing differentials of functions of stochastic processes are different from standard rules. These rules are provided by various forms of Ito's Lemma or change of variable formulas (CVF). Ito's Lemma is a "rule" of how to compute differentials when the basic source of uncertainty is Brownian motion. The CVF provides corresponding rules when uncertainty stems from Poisson processes or Levy processes.

10.2.1 Why all this?

"Computing differentials of functions of stochastic processes" sounds pretty abstract. Let us start with an example from deterministic continuous time setups which gives an idea about what the economic background for such differentials are.

Imagine the capital stock of an economy follows $\dot{K}(t) = I(t) - \delta K(t)$, an ordinary differential equation (ODE) known from ch. 4. Assume further that total factor productivity grows at an exogenous rate of g, $\dot{A}(t)/A(t) = g$. Let output be given by $Y(A(t), K(t), L)$ and let us ask how output grows over time. The reply would be provided by looking at the derivative of $Y(.)$ with respect to time,

$$\frac{d}{dt}Y(A(t), K(t), L) = Y_A \frac{dA(t)}{dt} + Y_K \frac{dK(t)}{dt} + Y_L \frac{dL}{dt}.$$

Alternatively, written as a differential, we would have

$$dY(A(t), K(t), L) = Y_A dA(t) + Y_K dK(t) + Y_L dL.$$

We can now insert equations describing the evolution of TFP and capital, $dA(t)$ and $dK(t)$, and take into account that employment L is constant. This gives

$$dY(A(t), K(t), L) = Y_A g A(t)\, dt + Y_K [I(t) - \delta K(t)]\, dt + 0.$$

Dividing by dt would give a differential equation that describes the growth of Y, i.e. $\dot{Y}(t)$.

The objective of the subsequent sections is to provide rules on how to compute differentials, of which $dY(A(t), K(t), L)$ is an example, in setups where $K(t)$ or $A(t)$ are described by stochastic DEs and not ordinary DEs as just used in this example.

10.2.2 Computing differentials for Brownian motion

We will now provide various versions of Ito's Lemma. For formal treatments, see the references in "further reading" at the end of this chapter.

- One stochastic process

Lemma 10.2.1 *Consider a function $F(t, x)$ of the diffusion process $x \in \mathbb{R}$ that is at least twice differentiable in x and once in t. The diffusion process is described by $dx(t) = a(x(t), z(t), t)\, dt + b(x(t), z(t), t)\, dz(t)$ as in (10.1.5). The differential dF reads*

$$dF = F_t dt + F_x dx + \frac{1}{2} F_{xx} (dx)^2 \tag{10.2.1}$$

where $(dx)^2$ is computed by using

$$dtdt = dtdz = dzdt = 0, \ dzdz = dt. \tag{10.2.2}$$

The "rules" in (10.2.2) can intuitively be understood by thinking about the "length of a graph" of a function, more precisely speaking about the total variation. For differentiable functions, the variation is finite. For Brownian motion, which is continuous but not differentiable, the variation goes to infinity. As there is a finite variation for differentiable functions, the quadratic (co)variations involving dt in (10.2.2) are zero. The quadratic variation of Brownian motion, however, cannot be neglected and is given by dt. For details, see the further-reading chapter 10.6 on "mathematical background".

Let us look at an example. Assume that $x(t)$ is described by a generalized Brownian motion as in (10.1.5). The square of dx is then given by

$$(dx)^2 = a^2(.)(dt)^2 + 2a(.)\,b(.)\,dtdz + b^2(.)(dz)^2 = b^2(.)\,dt,$$

where the last equality uses the "rules" from (10.2.2). The differential of $F(t,x)$ then reads

$$\begin{aligned} dF &= F_t dt + F_x a(.)\,dt + F_x b(.)\,dz + \frac{1}{2}F_{xx}b^2(.)\,dt \\ &= \left\{F_t + F_x a(.) + \frac{1}{2}F_{xx}b^2(.)\right\}dt + F_x b(.)\,dz. \end{aligned} \tag{10.2.3}$$

When we compare this differential with the "normal" one, we recognize familiar terms: The partial derivatives times deterministic changes, $F_t + F_x a(.)$, would appear also in circumstances where x follows a deterministic evolution. Put differently, for $b(.) = 0$ in (10.1.5), the differential dF reduces to $\{F_t + F_x a(.)\}\,dt$. Brownian motion therefore affects the differential dF in two ways: first, the stochastic term dz is added and second, maybe more "surprisingly", the deterministic part of dF is also affected through the quadratic term containing the second derivative F_{xx}.

- The lemma for many stochastic processes

This was the simple case of one stochastic process. Now consider the case of many stochastic processes. Think of the price of many stocks traded on the stock market. We then have the following

Lemma 10.2.2 *Consider the following set of stochastic differential equations,*

$$\begin{aligned} dx_1 &= a_1 dt + b_{11}dz_1 + ... + b_{1m}dz_m, \\ &\vdots \\ dx_n &= a_n dt + b_{n1}dz_1 + ... + b_{nm}dz_m. \end{aligned}$$

In matrix notation, they can be written as

$$dx = adt + bdz(t)$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \; b = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}, \; dz = \begin{pmatrix} dz_1 \\ \vdots \\ dz_m \end{pmatrix}.$$

Consider further a function $F(t,x)$ from $[0,\infty[\times R^n$ to R with time t and the n processes in x as arguments. Then

$$dF(t,x) = F_t dt + \Sigma_{i=1}^n F_{x_i} dx_i + \frac{1}{2} \Sigma_{i=1}^n \Sigma_{j=1}^n F_{x_i x_j} [dx_i dx_j] \tag{10.2.4}$$

where, as an extension to (10.2.2),

$$dtdt = dtdz_i = dz_i dt = 0 \text{ and } dz_i dz_j = \rho_{ij} dt. \tag{10.2.5}$$

When all z_i are mutually independent then $\rho_{ij} = 0$ for $i \neq j$ and $\rho_{ij} = 1$ for $i = j$. When two Brownian motions z_i and z_j are correlated, ρ_{ij} is the correlation coefficient between their increments dz_i and dz_j.

- An example with two stochastic processes

Let us now consider an example for a function $F(t,x,y)$ of two stochastic processes. As an example, assume that x is described by a generalized Brownian motion similar to (10.1.5), $dx = a(t,x,y)\,dt + b(t,x,y)\,dz_x$ and the stochastic process y is described by $dy = c(t,x,y)\,dt + g(t,x,y)\,dz_y$. Ito's Lemma (10.2.4) gives the differential dF as

$$dF = F_t dt + F_x dx + F_y dy + \frac{1}{2}\left[F_{xx}(dx)^2 + 2F_{xy} dx dy + F_{yy}(dy)^2\right] \tag{10.2.6}$$

Given the rule in (10.2.5), the squares and the product in (10.2.6) are

$$(dx)^2 = b^2(t,x,y)\,dt, \qquad (dy)^2 = g^2(t,x,y)\,dt, \qquad dxdy = \rho_{xy} b(.)\,g(.)\,dt,$$

where ρ_{xy} is the correlation coefficient of the two processes. More precisely, it is the correlation coefficient of the two normally distributed random variables that underlie the Wiener processes. The differential (10.2.6) therefore reads

$$\begin{aligned} dF &= F_t dt + a(.)\,F_x dt + b(.)\,F_x dz_x + c(.)\,F_y dt + g(.)\,F_y dz_y \\ &+ \frac{1}{2}\left[\left(F_{xx} b^2(.)\,dt + 2\rho_{xy} F_{xy} b(.)\,g(.)\,dt + F_{yy} g^2(.)\,dt\right)\right] \\ &= \left\{F_t + a(.)\,F_x + c(.)\,F_y + \frac{1}{2}\left[b^2(.)\,F_{xx} + 2\rho_{xy} b(.)\,g(.)\,F_{xy} + g^2(.)\,F_{yy}\right]\right\} dt \\ &+ b(.)\,F_x dz_x + g(.)\,F_y dz_y \end{aligned} \tag{10.2.7}$$

Note that this differential is almost simply the sum of the differentials of each stochastic process independently. The only term that is added is the term that contains the correlation coefficient. In other words, if the two stochastic processes were independent, the differential of a function of several stochastic processes equals the sum of the differential of each stochastic process individually.

- An example with one stochastic process and many Brownian motions

A second example stipulates a stochastic process $x(t)$ governed by $dx = u_1 dt + \Sigma_{i=1}^m v_i dz_i$. This corresponds to $n = 1$ in the lemma above. When we compute the square of dx, we obtain

$$(dx)^2 = (u_1 dt)^2 + 2u_1 dt \left[\Sigma_{i=1}^m v_i dz_i\right] + \left(\Sigma_{i=1}^m v_i dz_i\right)^2 = 0 + 0 + \left(\Sigma_{i=1}^m v_i dz_i\right)^2,$$

where the second equality uses (10.2.2). The differential of $F(t, x)$ therefore reads from (10.2.4)

$$\begin{aligned} dF(t,x) &= F_t dt + F_x \left[u_1 dt + \Sigma_{i=1}^m v_i dz_i\right] + \frac{1}{2} F_{xx} \left[\Sigma_{i=1}^m v_i dz_i\right]^2 \\ &= \{F_t + F_x u_1\} dt + \frac{1}{2} F_{xx} \left[\Sigma_{i=1}^m v_i dz_i\right]^2 + F_x \Sigma_{i=1}^m v_i dz_i. \end{aligned}$$

Computing the $\left[\Sigma_{i=1}^m v_i dz_i\right]^2$ term requires to take potential correlations into account. For any two uncorrelated increments dz_i and dz_j, $dz_i dz_j$ would from (10.2.5) be zero. When they are correlated, $dz_i dz_j = \rho_{ij} dt$ which includes the case of $dz_i dz_i = dt$.

10.2.3 Computing differentials for Poisson processes

When we consider the differential of a function of the variable that is driven by the Poisson process, we need to take the following CVFs into consideration.

- One stochastic process

Lemma 10.2.3 *Let there be a stochastic process $x(t)$ driven by Poisson uncertainty $q(t)$ described by the following stochastic differential equation*

$$dx(t) = a(.)\, dt + b(.)\, dq(t).$$

Consider the function $F(t, x)$. The differential of this function is

$$dF(t,x) = F_t dt + F_x a(.)\, dt + \{F(t, x + b(.)) - F(t,x)\}\, dq. \tag{10.2.8}$$

What was stressed before for Brownian motion is valid here as well: the functions $a(.)$ and $b(.)$ in the deterministic and stochastic part of this SDE can have as arguments any combinations of $q(t)$, $x(t)$ and t or can be simple constants.

The rule in (10.2.8) is very intuitive: the differential of a function is given by the "normal terms" and by a "jump term". The "normal terms" include the partial derivatives with respect to time t and x times changes per unit of time (1 for the first argument and $a(.)$ for x) times dt. Whenever the process q increases, x increases by the $b(.)$. The "jump term" therefore captures that the function $F(.)$ jumps from $F(t, x)$ to $F(t, \tilde{x}) = F(t, x + b(.))$.

- Two stochastic processes

Lemma 10.2.4 *Let there be two independent Poisson processes* q_x *and* q_y *driving two stochastic processes* $x(t)$ *and* $y(t)$,

$$dx = a(.)\,dt + b(.)\,dq_x, \qquad\qquad dy = c(.)\,dt + g(.)\,dq_y$$

and consider the function $F(x, y)$. *The differential of this function is*

$$dF(x,y) = \{F_x a(.) + F_y c(.)\}\,dt + \{F(x + b(.), y) - F(x,y)\}\,dq_x \\ + \{F(x, y + g(.)) - F(x,y)\}\,dq_y. \qquad (10.2.9)$$

Again, this "differentiation rule" consists of the "normal" terms and the "jump terms". As the function $F(.)$ depends on two arguments, the normal term contains two drift components, $F_x a(.)$ and $F_y c(.)$ and the jump term contains the effect of jumps in q_x and in q_y. Note that the dt term does not contain the time derivative $F_t(x, y)$ as in this example, $F(x, y)$ is assumed not to be a function of time and therefore $F_t(x, y) = 0$. In applications where $F(.)$ is a function of time, the $F_t(.)$ would, of course, have to be taken into consideration. Basically, (10.2.9) is just the "sum" of two versions of (10.2.8). There is no additional term as the correlation term in the case of Brownian motion in (10.2.7). This is due to the fact that any two Poisson processes are, by construction, independent.

Let us now consider a case that is frequently encountered in economic models when there is one economy-wide source of uncertainty, say new technologies arrive or commodity price shocks occur according to some Poisson process, and many variables in this economy (e.g. all relative prices) are affected simultaneously by this one shock. The CVF in situations of this type reads

Lemma 10.2.5 *Let there be two variables* x *and* y *following*

$$dx = a(.)\,dt + b(.)\,dq, \qquad\qquad dy = c(.)\,dt + g(.)\,dq,$$

where uncertainty stems from the same q *for both variables. Consider the function* $F(x, y)$. *The differential of this function is*

$$dF(x,y) = \{F_x a(.) + F_y c(.)\}\,dt + \{F(x + b(.), y + g(.)) - F(x,y)\}\,dq.$$

One nice feature about differentiation rules for Poisson processes is their very intuitive structure. When there are two independent Poisson processes as in (10.2.9), the change in F is given by either $F(x + b(.), y) - F(x, y)$ or by $F(x, y + g(.)) - F(x, y)$, depending on whether one or the other Poisson process jumps. When both arguments x and y are affected by the same Poisson process, the change in F is given by $F(x + b(.), y + g(.)) - F(x, y)$, i.e. the level of F after a simultaneous change of both x and y minus the pre-jump level $F(x, y)$.

- Many stochastic processes

We now present the most general case. Let there be n stochastic processes $x_i(t)$ and define the vector $x(t) = (x_1(t), ..., x_n(t))^T$. Let stochastic processes be described by n SDEs

$$dx_i(t) = \alpha_i(.)\, dt + \beta_{i1}(.)\, dq_1 + ... + \beta_{im}(.)\, dq_m, \quad i = 1, \ldots, n, \tag{10.2.10}$$

where $\beta_{ij}(.)$ stands for $\beta_{ij}(t, x(t))$. Each stochastic process $x_i(t)$ is driven by the same m Poisson processes. The impact of Poisson process q_j on $x_i(t)$ is captured by $\beta_{ij}(.)$. Note the similarity to the setup for the Brownian motion case in (10.2.4).

Proposition 10.2.1 *Let there be n stochastic processes described by (10.2.10). For a once continuously differentiable function $F(t, x)$, the process $F(t, x)$ obeys*

$$\begin{aligned} dF(t, x(t)) = &\{F_t(.) + \Sigma_{i=1}^n F_{x_i}(.)\, \alpha_i(.)\}\, dt \\ &+ \Sigma_{j=1}^m \left\{F\left(t, x(t) + \beta_j(.)\right) - F(t, x(t))\right\} dq_j, \end{aligned} \tag{10.2.11}$$

where F_t and F_{x_i}, $i = 1, \ldots, n$, denote the partial derivatives of f with respect to t and x_i, respectively, and β_j stands for the n-dimensional vector function $\left(\beta_{1j}, \ldots, \beta_{nj}\right)^T$.

The intuitive understanding is again simplified by focusing on "normal" continuous terms and on "jump terms". The continuous terms are as before and simply describe the impact of the $\alpha_i(.)$ in (10.2.10) on $F(.)$. The jump terms show how $F(.)$ changes from $F(t, x(t))$ to $F\left(t, x(t) + \beta_j(.)\right)$ when Poisson process j jumps. The argument $x(t)+\beta_j(.)$ after the jump of q_j is obtained by adding β_{ij} to component x_i in x, i.e. $x(t) + \beta_j(.) = \left(x_1 + \beta_{1j}, x_2 + \beta_{2j}, ..., x_n + \beta_{nj}\right)$.

10.2.4 Brownian motion and a Poisson process

There are much more general stochastic processes in the literature than just Brownian motion or Poisson processes. This section provides a CVF for a function of a variable which is driven by both Brownian motion and a Poisson process. More general processes than just additive combinations are so-called Levy processes, which will be analyzed in future editions of these notes.

Lemma 10.2.6 *Let there be a variable x which is described by*

$$dx = a(.)\, dt + b(.)\, dz + g(.)\, dq \tag{10.2.12}$$

and where uncertainty stems from Brownian motion z and a Poisson process q. Consider the function $F(t, x)$. The differential of this function is

$$dF(t, x) = \left\{F_t + F_x a(.) + \frac{1}{2} F_{xx} b^2(.)\right\} dt + F_x b(.)\, dz + \{F(t, x + g(.)) - F(t, x)\}\, dq. \tag{10.2.13}$$

Note that this lemma is just a "combination" of Ito's Lemma (10.2.3) and the CVF for a Poisson process from (10.2.8). For an arrival rate of zero, i.e. for $dq = 0$ at all times, (10.2.13) is identical to (10.2.3). For $b(.) = 0$, (10.2.13) is identical to (10.2.8).

10.3 Applications

10.3.1 Option pricing

One of the most celebrated papers in economics is the paper by Black and Scholes (1973) in which they derived a pricing formula for options. This section presents the first steps towards obtaining this pricing equation. The subsequent chapter 10.4.1 will complete the analysis. This section presents a simplified version (by neglecting jumps in the asset price) of the derivation of Merton (1976). The basic question is: what is the price of an option on an asset if there is absence of arbitrage on capital markets?

- The asset and option price

The starting point is the price S of an asset which evolves according to a geometric process

$$\frac{dS}{S} = \alpha dt + \sigma dz. \tag{10.3.1}$$

Uncertainty is modelled by the increment dz of Brownian motion. We assume that the economic environment is such (inter alia short selling is possible, there are no transaction costs) that the price of the option is given by a function $F(.)$ having as arguments only the price of the asset and time, $F(t, S(t))$. The differential of the price of the option is then given from (10.2.1) by

$$dF = F_t dt + F_S dS + \frac{1}{2} F_{SS} [dS]^2 . \tag{10.3.2}$$

As by (10.2.2) the square of dS is given by $(dS)^2 = \sigma^2 S^2 dt$, the differential reads

$$dF = \left\{ F_t + \alpha S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \right\} dt + \sigma S F_S dz \Leftrightarrow \frac{dF}{F} \equiv \alpha_F dt + \sigma_F dz \tag{10.3.3}$$

where the last step defined

$$\alpha_F = \frac{F_t + \alpha S F_S + \frac{1}{2} \sigma^2 S^2 F_{SS}}{F}, \quad \sigma_F = \frac{\sigma S F_S}{F}. \tag{10.3.4}$$

- Absence of arbitrage

Now comes the trick - the no-arbitrage consideration. Consider a portfolio that consists of N_1 units of the asset itself, N_2 options and N_3 units of some riskless assets, say wealth in a savings account. The price of such a portfolio is then given by

$$P = N_1 S + N_2 F + N_3 M,$$

where M is the price of one unit of the riskless asset. The proportional change of the price of this portfolio can be expressed as (holding the N_is constant, otherwise Ito's Lemma would have to be used)

$$dP = N_1 dS + N_2 dF + N_3 dM \Leftrightarrow \frac{dP}{P} = \frac{N_1 S}{P}\frac{dS}{S} + \frac{N_2 F}{P}\frac{dF}{F} + \frac{N_3 M}{P}\frac{dM}{M}.$$

Defining shares of the portfolio held in these three assets by $\theta_1 \equiv N_1 S/P$ and $\theta_2 \equiv N_2 F/P$, inserting option and stock price evolutions from (10.3.1) and (10.3.2) and letting the riskless asset M pay a constant return of r, we obtain

$$\begin{aligned} dP/P &\equiv \theta_1 \alpha dt + \theta_1 \sigma dz + \theta_2 \alpha_F dt + \theta_2 \sigma_F dz + (1 - \theta_1 - \theta_2)\, r dt \\ &= \{\theta_1 [\alpha - r] + \theta_2 [\alpha_F - r] + r\}\, dt + \{\theta_1 \sigma + \theta_2 \sigma_F\}\, dz. \end{aligned} \tag{10.3.5}$$

Now assume someone chooses weights such that the portfolio no longer bears any risk

$$\theta_1 \sigma + \theta_2 \sigma_F = 0. \tag{10.3.6}$$

The return of such a portfolio with these weights must then of course be identical to the return of the riskless interest asset, i.e. identical to r,

$$\left.\frac{dP/dt}{P}\right|_{riskless} = \theta_1 [\alpha - r] + \theta_2 [\alpha_F - r] + r|_{\theta_1 \sigma = -\theta_2 \sigma_F} = r \Leftrightarrow \frac{\alpha - r}{\sigma} = \frac{\alpha_F - r}{\sigma_F}.$$

If the return of the riskless portfolio did not equal the return of the riskless interest rates, there would be arbitrage possibilities. This approach is therefore called no-arbitrage pricing.

- The Black-Scholes formula

Finally, inserting α_F and σ_F from (10.3.4) yields the celebrated differential equation that determines the evolution of the price of the option,

$$\frac{\alpha - r}{\sigma} = \frac{F_t + \alpha S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} - rF}{\sigma S F_S} \Leftrightarrow \frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S - rF + F_t = 0. \tag{10.3.7}$$

Clearly, this equation does not to say what the price F of the option actually is. It only says how it changes over time and in reaction to S. But as we will see in ch. 10.4.1, this equation can actually be solved explicitly for the price of the option. Note also that we did not make any assumption so far about what type of option we are talking about.

10.3.2 Deriving a budget constraint

Most maximization problems require a constraint. For a household, this is usually the budget constraint. It is shown here how the structure of the budget constraint depends on the economic environment the household finds itself in and how the CVF needs to be applied here.

Let wealth at time t be given by the number $n(t)$ of stocks a household owns times their price $v(t)$, $a(t) = n(t)v(t)$. Let the price follow a process that is exogenous to the household (but potentially endogenous in general equilibrium),

$$dv(t) = \alpha v(t)\,dt + \beta v(t)\,dq(t), \tag{10.3.8}$$

where α and β are constants. For β we require $\beta > -1$ to avoid that the price can become zero or negative. Hence, the price grows with the continuous rate α and at discrete random times it jumps by β percent. The random times are modeled by the jump times of a Poisson process $q(t)$ with arrival rate λ, which is the "probability" that in the current period a price jump occurs. The expected (or average) growth rate is then given by $\alpha + \lambda\beta$ (see ch. 10.5.4).

Let the household earn dividend payments, $\pi(t)$ per unit of asset it owns, and labour income, $w(t)$. Assume furthermore that it spends $p(t)c(t)$ on consumption, where $c(t)$ denotes the consumption quantity and $p(t)$ the price of one unit of the consumption good. When buying stocks is the only way of saving, the number of stocks held by the household changes in a deterministic way according to

$$dn(t) = \frac{n(t)\pi(t) + w(t) - p(t)c(t)}{v(t)}dt.$$

When savings $n(t)\pi(t) + w(t) - p(t)c(t)$ are positive, the number of stocks held by the household increases by savings divided by the price of one stock. When savings are negative, the number of stocks decreases.

The change in the household's wealth, i.e. the household's budget constraint, is then given by applying the CVF to $a(t) = n(t)v(t)$. The appropriate CVF comes from (10.2.9) where only one of the two differential equations shows the increment of the Poisson process explicitly. With $F(x,y) = xy$, we obtain

$$\begin{aligned} da(t) &= \left\{ v(t)\frac{n(t)\pi(t) + w(t) - p(t)c(t)}{v(t)} + n(t)\alpha v(t) \right\} dt \\ &\quad + \{n(t)[v(t) + \beta v(t)] - n(t)v(t)\}\,dq(t) \\ &= \{r(t)a(t) + w(t) - p(t)c(t)\}\,dt + \beta a(t)\,dq(t), \end{aligned} \tag{10.3.9}$$

where the interest-rate is defined as

$$r(t) \equiv \frac{\pi(t)}{v(t)} + \alpha.$$

This is a very intuitive budget constraint: As long as the asset price does not jump, i.e., $dq(t) = 0$, the household's wealth increases by current savings, $r(t)a(t) + w(t) - p(t)c(t)$, where the interest rate, $r(t)$, consists of dividend payments in terms of the asset price plus the deterministic growth rate of the asset price. If a price jump occurs, i.e., $dq(t) = 1$, wealth jumps, as the price, by β percent, which is the stochastic part of the overall interest-rate. Altogether, the average interest rate amounts to $r(t) + \lambda\beta$ (see ch. 10.5.4).

10.4 Solving stochastic differential equations

Just as there are theorems on uniqueness and existence of solutions for ordinary differential equations, there are theorems for SDEs on these issues. There are also solution methods for SDEs. Here, we will consider some examples for solutions of SDEs.

Just as for ordinary deterministic differential equations in ch. 4.3.2, we will simply present solutions and not show how they can be derived. Solutions of stochastic differential equations $d(x(t))$ are, in analogy to the definition for ODE, again time paths $x(t)$ that satisfy the differential equation. Hence, by applying Ito's Lemma or the CVF, one can verify whether the solutions presented here are indeed solutions.

10.4.1 Some examples for Brownian motion

This section first looks at SDEs with Brownian motion which are similar to the ones that were presented when introducing SDEs in ch. 10.1.2: We start with Brownian motion with drift as in (10.1.3) and then look at an example for generalized Brownian motion in (10.1.5). In both cases, we work with SDEs which have an economic interpretation and are not just SDEs. Finally, we complete the analysis of the Black-Scholes option pricing approach.

- Brownian motion with drift 1

As an example for Brownian motion with drift, consider a representation of a production technology which could be called a "differential-representation" for the technology. This type of presenting technologies was dominant in early contributions that used continuous time methods under uncertainty but is sometimes still used today. A simple example is

$$dY(t) = AKdt + \sigma K dz(t), \tag{10.4.1}$$

where $Y(t)$ is output in t, A is a (constant) measure of total factor productivity, K is the (constant) capital stock, σ is some variance measure of output and z is Brownian motion. The change of output at each instant is then given by $dY(t)$. See "further reading" on references to the literature.

What does such a representation of output imply? To see this, look at (10.4.1) as Brownian motion with drift, i.e. consider A, K, and σ to be a constant. The solution to this differential equation starting in $t = 0$ with Y_0 and $z(0)$ is

$$Y(t) = Y_0 + AKt + \sigma K[z(t) - z(0)].$$

To simplify an economic interpretation set $Y_0 = z(0) = 0$. Output is then given by $Y(t) = (At + \sigma z(t))K$. This says that with a constant factor input K, output in t is determined by a deterministic and a stochastic part. The deterministic part At implies linear (i.e. not exponential as is usually assumed) growth, the stochastic part $\sigma z(t)$ implies deviations from the trend. As $z(t)$ is Brownian motion, the sum of the deterministic and stochastic part can become negative. This is an undesirable property of this approach.

To see that $Y(t)$ is in fact a solution of the above differential-representation, just apply Ito's Lemma and recover (10.4.1).

- Brownian motion with drift 2

As a second example and in an attempt to better understand why output can become negative, consider a standard representation of a technology $Y(t) = A(t)K$ and let TFP A follow Brownian motion with drift,

$$dA(t) = gdt + \sigma dz(t),$$

where g and σ are constants. What does this alternative specification imply?

Solving the SDE yields $A(t) = A_0 + gt + \sigma z(t)$ (which can again be checked by applying Ito's lemma). Output is therefore given by

$$Y(t) = (A_0 + gt + \sigma z(t))K = A_0 K + gKt + \sigma K z(t)$$

and can again become negative.

- Geometric Brownian motion

Let us now assume that TFP follows geometric Brownian motion process,

$$dA(t)/A(t) = gdt + \sigma dz(t), \tag{10.4.2}$$

where again g and σ are constants. Let output continue to be given by $Y(t) = A(t)K$.

The solution for TFP, provided an initial condition $A(0) = A_0$, is given by

$$A(t) = A_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t + \sigma z(t)}. \tag{10.4.3}$$

At any point in time t, the TFP level depends on time t and the current level of the stochastic process $z(t)$. This shows that TFP at each point t in time is random and thereby unknown from the perspective of $t = 0$. Hence, a SDE and its solution describe the deterministic evolution of a distribution over time. One could therefore plot a picture of $A(t)$ which in principle would look like the evolution of the distribution in ch. 7.4.1.

Interestingly, and this is due to the geometric specification in (10.4.2) and important for representing technologies in general, TFP can not become negative. While Brownian motion $z(t)$ can take any value between minus and plus infinity, the term $e^{\left(g - \frac{1}{2}\sigma^2\right)t + \sigma z(t)}$ is always positive. With an AK specification for output, output is always positive, $Y(t) = A_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t + \sigma z(t)} K$. In fact, it can be shown that output and TFP are lognormally distributed. Hence, the specification of TFP with geometric Brownian motion provides an alternative to the differential-representation in (10.4.1) which avoids the possibility of negative output.

The level of TFP at some future point in time t is determined by a deterministic part, $\left(g - \frac{1}{2}\sigma^2\right)t$, and by a stochastic part, $\sigma z(t)$. Apparently, the stochastic nature of TFP has an effect on the deterministic term. The structure (the factor 1/2 and the quadratic term σ^2) reminds of the role the stochastic disturbance plays in Ito's lemma. There as well (see e.g. (10.2.3)), the stochastic disturbance affects the deterministic component of the differential. As we will see later, however, this does not affect expected growth. In fact, we will show further below in (10.5.2) that expected output grows at the rate g (and is thereby independent of the variance parameter σ).

One can verify that (10.4.3) is a solution to (10.4.2) by using Ito's lemma. To do so, we need to bring (10.4.2) into a form which allows us to apply the formulas which we got to know in ch. 10.2.2. Define $x(t) \equiv \left(g - \frac{1}{2}\sigma^2\right)t + \sigma z(t)$ and $A(t) \equiv F(x(t)) = A_0 e^{x(t)}$. As a consequence, the differential for $x(t)$ is a nice SDE, $dx(t) = \left(g - \frac{1}{2}\sigma^2\right)dt + \sigma dz(t)$. As this SDE is of the form as in (10.1.5), we can use Ito's lemma from (10.2.3) and find

$$dA(t) = dF(x(t)) = \left\{F_x(x(t))\left[g - \frac{1}{2}\sigma^2\right] + \frac{1}{2}F_{xx}(x(t))\sigma^2\right\}dt + F_x(x(t))\sigma dz.$$

Inserting the first and second derivatives of $F(x(t))$ yields

$$\begin{aligned} dA(t) &= \left\{A_0 e^{x(t)}\left[g - \frac{1}{2}\sigma^2\right] + \frac{1}{2}A_0 e^{x(t)}\sigma^2\right\}dt + A_0 e^{x(t)}\sigma dz \Leftrightarrow \\ dA(t)/A(t) &= \left\{g - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\right\}dt + \sigma dz = g dt + \sigma dz, \end{aligned}$$

where the "iff" reinserted $A(t) = A_0 e^{x(t)}$ and divided by $A(t)$. As $A(t)$ from (10.4.3) satisfies the original SDE (10.4.2), $A(t)$ is a solution of (10.4.2).

- Option pricing

Let us come back to the Black-Scholes formula for option pricing. The SDE derived above in (10.3.7) describes the evolution of the price $F(t, S(t))$, where t is time and $S(t)$ the price of the underlying asset at t. We now look at a European call option, i.e. an option which gives the right to buy an asset at some fixed point in time T, the maturity date of the option. The fixed exercise or strike price of the option, i.e. the price at which the asset can be bought is denoted by P.

Clearly, at any point in time t when the price of the asset is zero, the value of the option is zero as well. This is the first boundary condition for our partial differential equation (10.3.7). When the option can be exercised at T and the price of the asset is S, the value of the option is zero if the exercise price P exceeds the price of the asset and $S - P$ if not. This is the second boundary condition.

$$F(t, 0) = 0, \quad F(T, S) = \max\{0, S - P\},$$

In the latter case where $S - P > 0$, the owner of the option would in fact buy the asset.

Given these two boundary conditions, the partial differential equation has the solution

$$F(t, S(t)) = S(t)\phi(d_1) - Pe^{-r[T-t]}\phi(d_2) \tag{10.4.4}$$

where

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du,$$

$$d_1 = \frac{\ln \frac{S(t)}{P} + \left(r \pm \frac{r^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

The expression $F(t, S(t))$ gives the price of an option at a point in time $t \leq T$ where the price of the asset is $S(t)$. It is a function of the cumulative standard normal distribution $\phi(y)$. For any path of $S(t)$, time up to maturity T affects the option price through d_1, d_2 and directly in the second term of the above difference. More interpretation is offered by many finance textbooks.

10.4.2 A general solution for Brownian motions

Consider the linear stochastic differential equation for $x(t)$,

$$dx(t) = \{a(t)x(t) + b(t)\} dt + \Sigma_{i=1}^{m} \{c_i(t)x(t) + g_i(t)\} dz^i(t) \tag{10.4.5}$$

where $a(t)$, $b(t)$, $c_i(t)$ and $g_i(t)$ are functions of time and $z^i(t)$ are Brownian motions. The correlation coefficients of its increment with the increments of $z^j(t)$ are ρ_{ij}. Let there be a boundary condition $x(0) = x_0$. The solution to (10.4.5) is

$$x(t) = e^{y(t)}\varphi(t) \tag{10.4.6}$$

where

$$y(t) = \int_0^t \left\{a(u) - \frac{1}{2}\Psi(u)\right\} du + \Sigma_{i=1}^{m} \int_0^t c_i(u)\, dz_i(u), \tag{10.4.7a}$$

$$\varphi(t) = x_0 + \int_0^t e^{-y(s)} \{b(s) - \Phi(s)\} ds + \Sigma_{i=1}^{m} \int_0^t e^{-y(s)} g_i(s)\, dz_i(s), \tag{10.4.7b}$$

$$\Psi(s) = \Sigma_{i=1}^{m}\Sigma_{j=1}^{m}\rho_{ij}c_i(s)\,c_j(s), \tag{10.4.7c}$$

$$\Phi(s) = \Sigma_{i=1}^{m}\Sigma_{j=1}^{m}\rho_{ij}c_i(s)\,g_i(s). \tag{10.4.7d}$$

To obtain some intuition for (10.4.6), we can first consider the case of certainty. For $c_i(t) = g_i(t) = 0$, (10.4.5) is a linear ODE and the solution is $x(t) = e^{\int_0^t a(u)du}\left[x_0 + \int_0^t e^{-\int_0^s a(u)du} b(s)\, ds\right]$. This corresponds to the results we know from ch. 4.3.2, see (4.3.7). For the general case, we now prove that (10.4.6) indeed satisfies (10.4.5).

In order to use Ito's Lemma, write the claim (10.4.6) as

$$x(t) = e^{y(t)}\varphi(t) \equiv f(y(t), \varphi(t)) \tag{10.4.8}$$

where

$$dy(t) = \left\{ a(t) - \frac{1}{2}\Psi(t) \right\} dt + \Sigma_{i=1}^{m} c_i(t)\, dz_i(t), \tag{10.4.9a}$$

$$d\varphi(t) = e^{-y(t)} \{ b(t) - \Phi(t) \} dt + \Sigma_{i=1}^{m} e^{-y(t)} g_i(t)\, dz_i(t). \tag{10.4.9b}$$

are the differentials of (10.4.7a) and (10.4.7b).

In order to compute the differential of $x(t)$ in (10.4.8), we have to apply the multidimensional Ito-Formula (10.2.6) where time is not an argument of $f(.)$. This gives

$$dx(t) = e^{y(t)}\varphi(t)\, dy(t) + e^{y(t)} d\varphi(t) + \frac{1}{2}\left\{ e^{y(t)}\varphi(t)\, [dy]^2 + 2e^{y(t)}[dydz] + 0 * [dz]^2 \right\}. \tag{10.4.10}$$

As $[dy]^2 = \left[\Sigma_{i=1}^{m} c_i(t)\, dz_i(t)\right]^2$ by (10.2.5) - all terms multiplied by dt equal zero - we obtain

$$[dy]^2 = \Sigma_{i=1}^{m} \Sigma_{j=1}^{m} \rho_{ij} c_i(t)\, c_j(t)\, dt. \tag{10.4.11}$$

Further, again by (10.2.5),

$$\begin{aligned} dydz &= \left(\Sigma_{i=1}^{m} c_i(t)\, dz_i(t) \right) \left(\Sigma_{i=1}^{m} e^{-y(t)} g_i(t)\, dz_i(t) \right) \\ &= e^{-y(t)} \Sigma_{i=1}^{m} \Sigma_{j=1}^{m} \rho_{ij} c_i(t)\, g_i(t)\, dt. \end{aligned} \tag{10.4.12}$$

Hence, reinserting (10.4.6), (10.4.9), (10.4.11) and (10.4.12) in (10.4.10) gives

$$\begin{aligned} dx(t) = {} & x(t) \left[\left\{ a(t) - \frac{1}{2}\Psi(t) \right\} dt + \Sigma_{i=1}^{m} c_i(t)\, dz_i(t) \right] \\ & + \{ b(t) - \Phi(t) \} dt + \Sigma_{i=1}^{m} g_i(t)\, dz_i(t) \\ & + \frac{1}{2} \left[x(t) \left[\Sigma_{i=1}^{m} \Sigma_{j=1}^{m} \rho_{ij} c_i(t)\, c_j(t)\, dt \right] + 2\Sigma_{i=1}^{m} \Sigma_{j=1}^{m} \rho_{ij} c_i(t)\, g_i(t)\, dt \right] \end{aligned}$$

Rearranging gives

$$\begin{aligned} dx(t) &= \{ x(t)a(t) + b(t) \} dt + \Sigma_{i=1}^{m} \{ x(t)c_i(t) + g_i(t) \} dz_i(t) \\ & \quad - \left\{ \frac{1}{2} x(t)\Psi(t) + \Phi(t) \right\} dt \\ & \quad + \left\{ \frac{1}{2} x(t) \Sigma_{i=1}^{m} \Sigma_{j=1}^{m} \rho_{ij} c_i(t)\, c_j(t) + \Sigma_{i=1}^{m} \Sigma_{j=1}^{m} \rho_{ij} c_i(t)\, g_i(t) \right\} dt \\ &= \{ x(t)a(t) + b(t) \} dt + \Sigma_{i=1}^{m} \{ x(t)c_i(t) + g_i(t) \} dz_i(t), \end{aligned}$$

where the last equality sign used (10.4.7c) and (10.4.7d). This is the original SDE in (10.4.5) which shows that the claim (10.4.6) is indeed a solution of (10.4.5).

10.4.3 Differential equations with Poisson processes

The presentation of solutions and their verification for Poisson processes follows a similar structure as for Brownian motion. We start here with a geometric Poisson process and compare properties of the solution with the TFP Brownian motion case. We then look at a more general process - the description of a budget constraint - which extends the geometric Poisson process. One should keep in mind, as stressed already in ch. 4.3.3, that solutions to differential equations are different from the integral representation of e.g. ch. 10.1.3.

- A geometric Poisson process

Imagine that TFP follows a deterministic trend and occasionally makes a discrete jump. This is captured by a geometric description as in (10.4.2), only that Brownian motion is replaced by a Poisson process,

$$dA(t)/A(t) = gdt + \sigma dq(t). \tag{10.4.13}$$

Again, g and σ are constant with $\sigma > -1$.

The solution to this SDE is given by

$$A(t) = A_0 e^{gt + [q(t) - q(0)] \ln(1+\sigma)}. \tag{10.4.14}$$

Uncertainty does not affect the deterministic part here, in contrast to the solution (10.4.3) for the Brownian motion case. As before, TFP follows a deterministic growth component and a stochastic component, $[q(t) - q(0)] \ln(1+\sigma)$. The latter makes future TFP uncertain from the perspective of today.

The claim that $A(t)$ is a solution can be proven by applying the appropriate CVF. This will be done for the next, more general, example.

- A budget constraint

As a second example, we look at a dynamic budget constraint,

$$da(t) = \{r(t) a(t) + w(t) - c(t)\} dt + \beta a(t) dq. \tag{10.4.15}$$

Defining $\Phi(s) \equiv w(s) - c(s)$, the backward solution of (10.4.15) with initial condition $a(0) = a_0$ reads

$$a(t) = e^{y(t)} \left[a_0 + \int_0^t e^{-y(s)} \Phi(s) ds \right] \tag{10.4.16}$$

where $y(t)$ is

$$y(t) = \int_0^t r(u) du + [q(t) - q(0)] \ln(1+\beta). \tag{10.4.17}$$

Note that the solution in (10.4.16) has the same structure as the solution to a deterministic version of the differential equation (10.4.15) (which we would obtain for $\beta = 0$). In fact, the structure of (10.4.16) is identical to the structure of (4.3.7). Put differently, the stochastic component in (10.4.15) only affects the discount factor $y(t)$. This is not surprising - in a way - as uncertainty is proportional to $a(t)$ and the factor β can be seen as the stochastic component of the interest payments on $a(t)$.

We have just stated that (10.4.16) is a solution. This should therefore be verified. To this end, define

$$z(t) \equiv a_0 + \int_0^t e^{-y(s)}\Phi(s)ds, \tag{10.4.18}$$

and write the solution (10.4.16) as $a(t) = e^{y(t)}z(t)$ where from (10.4.17) and (10.4.18) and Leibniz' rule (4.3.3),

$$dy(t) = r(t)dt + \ln(1+\beta)dq(t), \qquad dz(t) = e^{-y(t)}\Phi(t)dt. \tag{10.4.19}$$

We have thereby defined a function $a(t) = F(y(t), z(t))$ where the SDEs describing the evolution of $y(t)$ and $z(t)$ are given in (10.4.19). This allows us to use the CVF (10.2.9) which then says

$$\begin{aligned} dF &= F_y r(t)\,dt + F_z e^{-y(t)}\Phi(t)\,dt + \{F(y+\ln(1+\beta), z) - F(y,z)\}\,dq \Leftrightarrow \\ da(t) &= e^{y(t)}z(t)r(t)dt + e^{y(t)}e^{-y(t)}\Phi(t)dt + \left\{e^{y(t)+\ln(1+\beta)}z(t) - e^{y(t)}z(t)\right\}dq \\ &= \{r(t)\,a(t) + \Phi(t)\}\,dt + \beta a(t)\,dq. \end{aligned}$$

This is the original differential equation. Hence, $a(t)$ in (10.4.16) is a solution for (10.4.15).

- The intertemporal budget constraint

In stochastic worlds, there is also a link between dynamic and intertemporal budget constraints, just as in deterministic setups as in ch. 4.4.2. We can now use the solution (10.4.16) to obtain an intertemporal budget constraint. We first present here a budget constraint for a finite planning horizon and then generalize the result.

For the finite horizon case, we can rewrite (10.4.16) as

$$\int_0^t e^{-y(s)}c(s)\,ds + e^{-y(t)}a(t) = a_0 + \int_0^t e^{-y(s)}w(s)\,ds. \tag{10.4.20a}$$

This formulation suggests a standard economic interpretation. Total expenditure over the planning horizon from 0 to t on the left-hand side must equal total wealth on the right-hand side. Total expenditure consists of the present value of the consumption expenditure path $c(s)$ and the present value of assets $a(t)$ the household wants to hold at the end of the planning period, i.e. at t. Total wealth is given by initial financial wealth a_0 and the present value of current and future wage income $w(s)$. All discounting takes place at the realized stochastic interest rate $y(s)$ - no expectations are formed.

In order to obtain an infinite horizon intertemporal budget constraint, the solution (10.4.16) should be written more generally - after replacing t by τ and 0 by t - as

$$a(\tau) = e^{y(\tau)}a_t + \int_t^\tau e^{y(\tau)-y(s)}(w(s) - c(s))\,ds \tag{10.4.21}$$

where $y(\tau)$ is

$$y(\tau) = \int_t^\tau r(u)\,du + \ln(1+\beta)\,[q(\tau) - q(t)]\,. \qquad (10.4.22)$$

Multiplying (10.4.21) by $e^{-y(\tau)}$ and rearranging gives

$$a(\tau)\,e^{-y(\tau)} + \int_t^\tau e^{-y(s)}(c(s) - w(s))ds = a(t).$$

Letting τ go to infinity, assuming a no-Ponzi game condition $\lim_{\tau\to\infty} a(\tau)\,e^{-y(\tau)} = 0$ and rearranging again yields

$$\int_t^\infty e^{-y(s)}c(s)\,ds = a(t) + \int_t^\infty e^{-y(s)}w(s)\,ds.$$

The structure here and in (10.4.20a) is close to the one in the deterministic world. The present value of consumption expenditure needs to equal current financial wealth $a(t)$ plus "human wealth", i.e. the present value of current and future labour income. Here as well, discounting takes place at the "risk-corrected" interest rate as captured by $y(\tau)$ in (10.4.22). Note also that this intertemporal budget constraint requires equality in realizations, not in expected terms.

- A switch process

Here are two funny processes which have an interesting and simple solution. Consider the initial value problem,

$$dx(t) = -2x(t)\,dq(t)\,, \quad x(0) = x_0.$$

The solution is $x(t) = (-1)^{q(t)}\,x_0$, i.e. $x(t)$ oscillates between $-x_0$ and x_0.

Now consider the transformation $y = \bar{y} + x$. It evolves according to

$$dy = -2(y - \bar{y})\,dq, \quad y_0 = \bar{y} + x_0.$$

Its solution is $y(t) = \bar{y} + (-1)^{q(t)}\,x_0$ and $y(t)$ oscillates now between $\bar{y} - x_0$ and $\bar{y} + x_0$.

This could be nicely used for models with wage uncertainty, where wages are sometimes high and sometimes low. An example would be a matching model where labour income is w (i.e. $\bar{y} + x_0$) when employed and b (i.e. $\bar{y} - x_0$) when unemployed. The difference between labour income levels is then $2x_0$. The drawback is that the probability of finding a job is identical to the probability of losing it.

10.5 Expectation values

10.5.1 The idea

What is the idea behind expectations of stochastic processes? When thinking of a stochastic process $X(t)$, either a simple one like Brownian motion or a Poisson process or more complex ones described by SDEs, it is useful to keep in mind that one can think of the value $X(\tau)$ the stochastic process takes at some future point τ in time as a "normal" random variable. At each future point in time $X(\tau)$ is characterized by some mean, some variance, by a range etc. This is illustrated in the following figure.

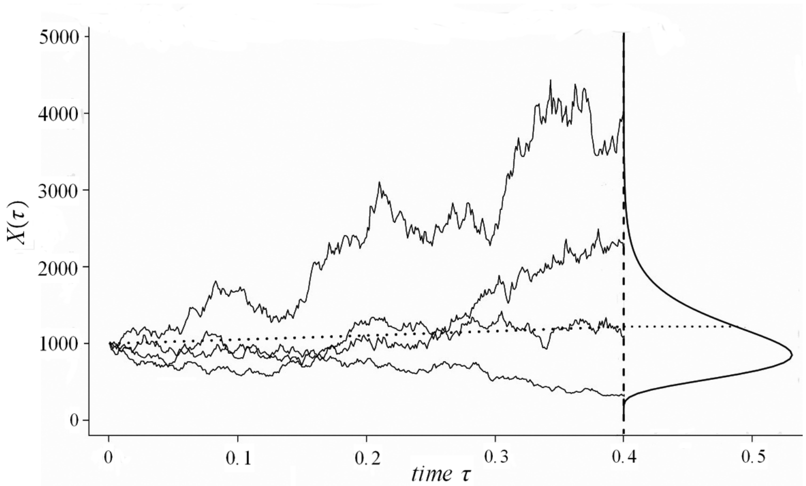


Figure 10.5.1 *The distribution of* $X(\tau)$ *at* $\tau = .4 > t = 0$

The figure shows four realizations of the stochastic process $X(t)$. The starting point is today in $t = 0$ and the mean growth of this process is illustrated by the dotted line. When we think of possible realizations of $X(\tau)$ for $\tau = .4$ from the perspective of $t = 0$, we can imagine a vertical line that cuts through possible paths at $\tau = .4$. With sufficiently many realizations of these paths, we would be able to make inferences about the distributional properties of $X(.4)$. As the process used for this simulation is a geometric Brownian motion process as in (10.4.2), we would find that $X(.4)$ is lognormally distributed as depicted above. Clearly, given that we have a precise mathematical description of our process, we do not need to estimate distributional properties for $X(\tau)$, as suggested by this figure, but we can explicitly compute them.

What this section therefore does is provide tools that allow to determine properties of the distribution of a stochastic process for future points in time. Put differently, understanding a stochastic process means understanding the evolution of its distributional properties. We will first start by looking at relatively straightforward ways to compute means. Subsequently, we provide some martingale results which allow us to then understand a more general approach to computing means and also higher moments.

10.5.2 Simple results

- Brownian motion I

Consider a Brownian motion process $Z(\tau), \tau \geq 0$. Let $Z(0) = 0$. From the definition of Brownian motion in ch. 10.1.1, we know that $Z(\tau)$ is normally distributed with mean 0 and variance $\sigma^2\tau$. Let us assume we are in t today and $Z(t) \neq 0$. What is the mean and variance of $Z(\tau)$ for $\tau > t$?

We construct a stochastic process $Z(\tau) - Z(t)$ which at $\tau = t$ is equal to zero. Now imagining that t is equal to zero, $Z(\tau) - Z(t)$ is a Brownian motion process as defined in ch. 10.1.1. Therefore, $E_t[Z(\tau) - Z(t)] = 0$ and $\text{var}_t[Z(\tau) - Z(t)] = \sigma^2[\tau - t]$. This says that the increments of Brownian motion are normally distributed with mean zero and variance $\sigma^2[\tau - t]$. Hence, the reply to our question is

$$E_t Z(\tau) = Z(t), \quad \text{var}_t Z(\tau) = \sigma^2[\tau - t].$$

For our convention $\sigma \equiv 1$ which we follow here as stated at the beginning of ch. 10.1.2, the variance would equal $\text{var}_t Z(\tau) = \tau - t$.

- Brownian motion II

Consider again the geometric Brownian process $dA(t)/A(t) = gdt + \sigma dz(t)$ describing the growth of TFP in (10.4.2). Given the solution $A(t) = A_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t + \sigma z(t)}$ from (10.4.3), we would now like to know what the expected TFP level $A(t)$ in t from the perspective of $t = 0$ is. To this end, apply the expectations operator E_0 and find

$$E_0 A(t) = A_0 E_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t + \sigma z(t)} = A_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t} E_0 e^{\sigma z(t)}, \tag{10.5.1}$$

where the second equality exploited the fact that $e^{\left(g - \frac{1}{2}\sigma^2\right)t}$ is non-random. As $z(t)$ is a standardized Brownian motion, $z(t) \sim N(0, t)$, $\sigma z(t)$ is normally distributed with $N(0, \sigma^2 t)$. As a consequence (compare ch. 7.3.3, especially eq. (7.3.4)), $e^{\sigma z(t)}$ is lognormally distributed with mean $e^{\frac{1}{2}\sigma^2 t}$. Hence,

$$E_0 A(t) = A_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t} e^{\frac{1}{2}\sigma^2 t} = A_0 e^{gt}. \tag{10.5.2}$$

The expected level of TFP growing according to a geometric Brownian motion process grows at the drift rate g of the stochastic process. The variance parameter σ does not affect expected growth.

Note that we can also determine the variance of $A(t)$ by simply applying the variance operator to the solution from (10.4.3),

$$\text{var}_0 A(t) = \text{var}_0 \left(A_0 e^{\left(g - \frac{1}{2}\sigma^2\right)t} e^{\sigma z(t)}\right) = A_0^2 e^{2\left(g - \frac{1}{2}\sigma^2\right)t} \text{var}_0 \left(e^{\sigma z(t)}\right).$$

We now make similar arguments to before when we derived (10.5.2). As $\sigma z(t) \sim N(0, \sigma^2 t)$, the term $e^{\sigma z(t)}$ is lognormally distributed with, see (7.3.4), variance $e^{\sigma^2 t}\left(e^{\sigma^2 t}-1\right)$. Hence, we find

$$\begin{aligned} \mathrm{var}_0 A(t) &= A_0^2 e^{2\left(g-\frac{1}{2}\sigma^2\right)t} e^{\sigma^2 t}\left(e^{\sigma^2 t}-1\right) = A_0^2 e^{\left(2\left(g-\frac{1}{2}\sigma^2\right)+\sigma^2\right)t}\left(e^{\sigma^2 t}-1\right) \\ &= A_0^2 e^{2gt}\left(e^{\sigma^2 t}-1\right). \end{aligned}$$

This clearly shows that for any $g > 0$, the variance of $A(t)$ increases over time.

- Poisson processes I

The expected value and variance of a Poisson process $q(\tau)$ with arrival rate λ are

$$\begin{aligned} E_t q(\tau) &= q(t) + \lambda[\tau - t], \qquad \tau > t \\ \mathrm{var}_t q(\tau) &= \lambda[\tau - t]. \end{aligned} \tag{10.5.3}$$

As always, we compute moments from the perspective of t and $\tau > t$ lies in the future. The expected value $E_t q(\tau)$ directly follows from the definition of a Poisson process in ch. 10.1.1, see (10.1.1). As the number of jumps is Poisson distributed with mean $\lambda[\tau - t]$ and the variance of the Poisson distribution is identical to its mean, the variance also follows from the definition of a Poisson process. Note that there are simple generalizations of the Poisson process where the variance differs from the mean (see ch. 10.5.4).

- Poisson processes II

Poisson processes are widely used to understand events like finding a job, getting fired, developing a new technology, occurrence of an earthquake etc. One can model these situations by a Poisson process $q(t)$ with arrival rate λ. In these applications, the following questions are often asked.

What is the probability that $q(t)$ will jump exactly once between t and τ? Given that by def. 10.1.6, $\frac{e^{-\lambda[\tau-t]}(\lambda[\tau-t])^n}{n!}$ is the probability that q jumped n times by τ, i.e. $q(\tau) = q(t) + n$, this probability is given by $P(q(\tau) = q(t) + 1) = e^{-\lambda[\tau-t]}\lambda[\tau - t]$. Note that this is a function which is non-monotonic in time τ, as illustrated in fig. 10.5.2.

How can the expression $P(q(\tau) = q(t) + 1) = e^{-\lambda[\tau-t]}\lambda[\tau - t]$ be reconciled with the usual description of a Poisson process, as e.g. in (10.1.2), where it says that the probability of a jump is given by λdt? When we look at a small time interval dt, we can neglect $e^{-\lambda dt}$ as it is "very close to one" and we get $P(dq(t) = 1) = \lambda dt$. As over a very small instant dt a Poisson process can either jump or not jump (it can not jump more than once in a small instant dt), the probability of no jump is therefore $P(dq(t) = 0) = 1 - \lambda dt$.

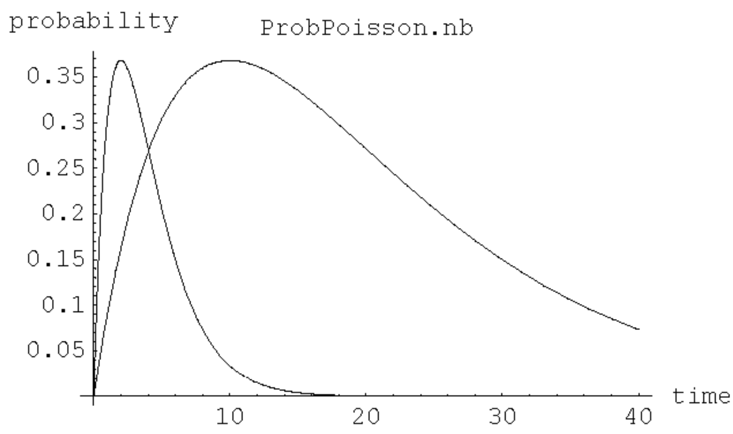


Figure 10.5.2 *The probability of one jump by time τ for a high arrival rate (peak at around 3) and a low arrival rate (peak at around 11)*

What is the length of time between two jumps or events? Clearly, we do not know as jumps are random. We can therefore only ask what the distribution of the length of time between two jumps is. To understand this, we start from (10.1.1) which tells us that the probability that there is no jump by τ is given by $P[q(\tau) = q(t)] = e^{-\lambda[\tau-t]}$. The probability that there is at least one jump by τ is therefore $P[q(\tau) > q(t)] = 1 - e^{-\lambda[\tau-t]}$. This term $1 - e^{-\lambda[\tau-t]}$ is the cumulative density function of an exponential distribution for a random variable $\tau - t$ (see ex. 2). Hence, the length $\tau - t$ between two jumps is exponentially distributed with density $\lambda e^{-\lambda[\tau-t]}$.

- Poisson process III

Following the same structure as with Brownian motion, we now look at the geometric Poisson process describing TFP growth (10.4.13). The solution was given in (10.4.14) which reads, slightly generalized with the perspective of t and initial condition A_t, $A(\tau) = A_t e^{g[\tau-t]+[q(\tau)-q(t)]\ln(1+\sigma)}$. What is the expected TFP level in τ?

Applying the expectation operator gives

$$E_t A(\tau) = A_t e^{g[\tau-t]-q(t)\ln(1+\sigma)} E_t e^{q(\tau)\ln(1+\sigma)} \tag{10.5.4}$$

where, as in (10.5.1), we split the exponential growth term into its deterministic and stochastic part. To proceed, we need the following

Lemma 10.5.1 *(Posch and Wälde, 2006) Assume that we are in t and form expectations about future arrivals of the Poisson process. The expected value of $c^{kq(\tau)}$, conditional on t where $q(t)$ is known, is*

$$E_t(c^{kq(\tau)}) = c^{kq(t)} e^{(c^k-1)\lambda(\tau-t)}, \quad \tau > t, \quad c, k = const.$$

Note that for integer k, these are the raw moments of $c^{q(\tau)}$.

Proof. We can trivially rewrite $c^{kq(\tau)} = c^{kq(t)}c^{k[q(\tau)-q(t)]}$. At time t, we know the realization of $q(t)$ and therefore $E_t c^{kq(\tau)} = c^{kq(t)} E_t c^{k[q(\tau)-q(t)]}$. Computing this expectation requires the probability that a Poisson process jumps n times between t and τ. Formally,

$$
\begin{aligned}
E_t c^{k[q(\tau)-q(t)]} &= \Sigma_{n=0}^{\infty} c^{kn} \frac{e^{-\lambda(\tau-t)}(\lambda(\tau-t))^n}{n!} = \Sigma_{n=0}^{\infty} \frac{e^{-\lambda(\tau-t)}(c^k\lambda(\tau-t))^n}{n!} \\
&= e^{(c^k-1)\lambda(\tau-t)} \Sigma_{n=0}^{\infty} \frac{e^{-\lambda(\tau-t)-(c^k-1)\lambda(\tau-t)}(c^k\lambda(\tau-t))^n}{n!} \\
&= e^{(c^k-1)\lambda(\tau-t)} \Sigma_{n=0}^{\infty} \frac{e^{-c^k\lambda(\tau-t)}(c^k\lambda(\tau-t))^n}{n!} = e^{(c^k-1)\lambda(\tau-t)},
\end{aligned}
$$

where $\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}$ is the probability of $q(\tau) = n$ and $\Sigma_{n=0}^{\infty} \frac{e^{-c^k\lambda(\tau-t)}(c^k\lambda(\tau-t))^n}{n!} = 1$ denotes the summation of the probability function over the whole support of the Poisson distribution which was used in the last step. For a generalization of this lemma, see ex. 8. ■

To apply this lemma to our case $E_t e^{q(\tau)\ln(1+\sigma)}$, we set $c = e$ and $k = \ln(1+\sigma)$. Hence, $E_t e^{q(\tau)\ln(1+\sigma)} = e^{\ln(1+\sigma)q(t)} e^{(e^{\ln(1+\sigma)}-1)\lambda[\tau-t]} = e^{q(t)\ln(1+\sigma)} e^{\sigma\lambda[\tau-t]} = e^{q(t)\ln(1+\sigma)+\sigma\lambda[\tau-t]}$ and, inserted in (10.5.4), the expected TFP level is

$$
E_t A(\tau) = A_t e^{g[\tau-t]-q(t)\ln(1+\sigma)} e^{q(t)\ln(1+\sigma)+\sigma\lambda[\tau-t]} = A_t e^{(g+\sigma\lambda)[\tau-t]}.
$$

This is an intuitive result: the growth rate of the expected TFP level is driven both by the deterministic growth component of the SDE (10.4.13) for TFP and by the stochastic part. The growth rate of expected TFP is higher, the higher the determinist part, the g, the more often the Poisson process jumps on average (a higher arrival rate λ) and the higher the jump (the higher σ).

This confirms formally what was already visible in and informally discussed after fig. 10.1.2 of ch. 10.1.2: A Poisson process as a source of uncertainty in a SDE implies that average growth is not just determined by the deterministic part of the SDE (as is the case when Brownian motion constitutes the disturbance term) but also by the Poisson process itself. For a positive σ, average growth is higher, for a negative σ, it is lower.

10.5.3 Martingales

Martingale is an impressive word for a simple concept. Here is a simplified definition which is sufficient for our purposes. For more complete definitions (in a technical sense), see "further reading".

Definition 10.5.1 *A stochastic process $x(t)$ is a martingale if, being in t today, the expected value of x at some future point τ in time equals the current value of x,*

$$
E_t x(\tau) = x(t), \quad \tau \geq t. \tag{10.5.5}
$$

As the expected value of $x(t)$ is $x(t)$, $E_t x(t) = x(t)$, this can easily be rewritten as $E_t[x(\tau) - x(t)] = 0$. This is identical to saying that $x(t)$ is a martingale if the expected value of its increments between now t and somewhere at τ in the future is zero. This definition and the definition of Brownian motion imply that Brownian motion is a martingale.

In what follows, we will use the martingale properties of certain processes relatively frequently, for example when computing moments. Here are now some fundamental examples for martingales.

- Brownian motion

First, look at Brownian motion, where we have a central result useful for many applications where expectations and other moments are computed. It states that $\int_t^\tau f(z(s), s)\, dz(s)$, where $f(.)$ is some function and $z(s)$ is Brownian motion, is a martingale (Corollary 3.2.6, Øksendal, 1998, p. 33). Hence,

$$E_t \int_t^\tau f(z(s), s)\, dz(s) = 0. \tag{10.5.6}$$

- Poisson uncertainty

A similar fundamental result for Poisson processes exists. We will use in what follows the martingale property of various expressions containing Poisson uncertainty. These expressions are identical to or special cases of $\int_t^\tau f(q(s), s)\, dq(s) - \lambda \int_t^\tau f(q(s), s)\, ds$, of which Garcia and Griego (1994, theorem 5.3) have shown that it is a martingale indeed. Hence,

$$E_t \left[\int_t^\tau f(q(s), s)\, dq(s) - \lambda \int_t^\tau f(q(s), s)\, ds \right] = 0. \tag{10.5.7}$$

As always, λ is the (constant) arrival rate of $q(s)$.

10.5.4 A more general approach to computing moments

When we want to understand moments of some stochastic process, we can proceed in two ways. Either, a SDE is expressed in its integral version, expectations operators are applied and the resulting deterministic differential equation is solved. Or, the SDE is solved directly and then expectation operators are applied. We already saw examples for the second approach in ch. 10.5.2. We will now follow the first way as this is generally the more flexible one. We first start with examples for Brownian motion processes and then look at cases with Poisson uncertainty.

- The drift and variance rates for Brownian motion

We will now return to our first SDE in (10.1.3), $dx(t) = adt + bdz(t)$, and want to understand why a is called the drift term and b^2 the variance term. To this end, we compute the mean and variance of $x(\tau)$ for $\tau > t$.

We start by expressing (10.1.3) in its integral representation as in ch. 10.1.3. This gives

$$x(\tau) - x(t) = \int_t^\tau ads + \int_t^\tau bdz(s) = a[\tau - t] + b[z(\tau) - z(t)]. \tag{10.5.8}$$

The expected value $E_t x(\tau)$ is then simply

$$E_t x(\tau) = x(t) + a[\tau - t] + bE_t[z(\tau) - z(t)] = x(t) + a[\tau - t], \tag{10.5.9}$$

where the second step used the fact that the expected increment of Brownian motion is zero. As the expected value $E_t x(\tau)$ is a deterministic variable, we can compute the usual time derivative and find how the expected value of $x(\tau)$, being today in t, changes the further the point τ lies in the future, $dE_t x(\tau)/d\tau = a$. This makes clear why a is referred to as the drift rate of the random variable $x(.)$.

Let us now analyse the variance of $x(\tau)$ where we also start from (10.5.8). The variance can from (7.3.1) be written as $\text{var}_t x(\tau) = E_t x^2(\tau) - [E_t x(\tau)]^2$. In contrast to the term in (7.3.1) we need to condition the variance on t: If we computed the variance of $x(\tau)$ from the perspective of some earlier t, the variance would differ - as will become very clear from the expression we will see in a second. Applying the expression from (7.3.1) also shows that we can look at any $x(\tau)$ as a "normal" random variable: Whether $x(\tau)$ is described by a stochastic process or by some standard description of a random variable, an $x(\tau)$ for any fix future point τ in time has some distribution with corresponding moments. This allows us to use standard rules for computing moments. Computing first $E_t x^2(\tau)$ gives by inserting (10.5.8)

$$\begin{aligned}
&E_t x^2(\tau) \\
=\;& E_t\left\{[x(t) + a[\tau - t] + b[z(\tau) - z(t)]]^2\right\} \\
=\;& E_t\left\{[x(t) + a[\tau - t]]^2 + 2[x(t) + a(\tau - t)]b[z(\tau) - z(t)] + [b(z(\tau) - z(t))]^2\right\} \\
=\;& [x(t) + a[\tau - t]]^2 + 2[x(t) + a(\tau - t)]bE_t[z(\tau) - z(t)] + E_t\left\{[b(z(\tau) - z(t))]^2\right\} \\
=\;& [x(t) + a[\tau - t]]^2 + b^2 E_t\left\{[z(\tau) - z(t)]^2\right\},
\end{aligned} \tag{10.5.10}$$

where the last equality used again that the expected increment of Brownian motion is zero. As $[E_t x(\tau)]^2 = [x(t) + a[\tau - t]]^2$ from (10.5.9), inserting (10.5.10) into the variance expression gives $\text{var}_t x(\tau) = b^2 E_t\left\{[z(\tau) - z(t)]^2\right\}$.

Computing the mean of the second moment gives

$$E_t\left\{[z(\tau) - z(t)]^2\right\} = E_t\left\{z^2(\tau) - 2z(\tau)z(t) + z^2(t)\right\} = E_t\left\{z^2(\tau)\right\} - z^2(t) = \text{var}_t z(\tau),$$

where we used that $z(t)$ is known in t and therefore non-stochastic, that $E_t z(\tau) = z(t)$ and equation (7.3.1) again. We therefore found that $\text{var}_t x(\tau) = b^2 \text{var}_t z(\tau)$, the variance of $x(\tau)$ is b^2 times the variance of $z(\tau)$. As the latter variance is given by $\text{var}_t z(\tau) = \tau - t$, given that we focus on Brownian motion with standard normally distributed increments, we found

$$var_t x(\tau) = b^2 [\tau - t].$$

This equation shows why b is the variance parameter for $x(.)$ and why b^2 is called its variance rate. The expression also makes clear why it is so important to state the current point in time, t in our case. If we were further in the past or τ is further in the future, the variance would be larger.

- Expected returns - Brownian motion

Imagine an individual owns wealth a that is allocated to N assets such that shares in wealth are given by $\theta_i \equiv a_i/a$. The price of each asset follows a certain pricing rule, say geometric Brownian motion, and let's assume that total wealth of the household, neglecting labour income and consumption expenditure, evolves according to

$$da(t) = a(t)\left[rdt + \Sigma_{i=1}^N \theta_i \sigma_i dz_i(t)\right], \tag{10.5.11}$$

where $r \equiv \Sigma_{i=1}^N \theta_i r_i$. This is in fact the budget constraint (with $w - c = 0$) which will be derived and used in ch. 11.4 on capital asset pricing. Note that Brownian motions z_i are correlated, i.e. $dz_i dz_j = \rho_{ij} dt$ as in (10.2.5). What is the expected return and the variance of holding such a portfolio, taking θ_i and interest rates and variance parameters as given?

Using the same approach as in the previous example we find that the expected return is simply given by r. This is due to the fact that the mean of the BMs $z_i(t)$ are zero. The variance of $a(t)$ is left as an exercise.

- Expected returns - Poisson

Let us now compute the expected return of wealth when the evolution of wealth is described by the budget constraint in (10.3.9), $da(t) = \{r(t)a(t) + w(t) - p(t)c(t)\}dt + \beta a(t)dq(t)$. When we want to do so, we first need to be precise about what we mean by the expected return. We define it as the growth rate of the mean of wealth when consumption expenditure and labour income are identical, i.e. when total wealth changes only due to capital income.

Using this definition, we first compute the expected wealth level at some future point in time τ. Expressing this equation in its integral representation as in ch. 10.1.3 gives

$$a(\tau) - a(t) = \int_t^\tau \{r(s)a(s) + w(s) - p(s)c(s)\}ds + \int_t^\tau \beta a(s)dq(s).$$

Applying the expectations operator yields

$$
\begin{aligned}
E_t a(\tau) - a(t) &= E_t \int_t^\tau \{r(s)\, a(s) + w(s) - p(s)\, c(s)\}\, ds + E_t \int_t^\tau \beta a(s)\, dq(s) \\
&= \int_t^\tau E_t r(s)\, E_t a(s)\, ds + \beta\lambda \int_t^\tau E_t a(s)\, ds. \qquad (10.5.12)
\end{aligned}
$$

The second equality used an element of the definition of the expected return, i.e. $w(s) = p(s)\, c(s)$, that $r(s)$ is independent of $a(s)$ and the martingale result of (10.5.7). When we define the mean of $a(s)$ from the perspective of t by $\mu(s) \equiv E_t a(s)$, this equation reads

$$
\mu(\tau) - a(t) = \int_t^\tau E_t r(s)\, \mu(s)\, ds + \beta\lambda \int_t^\tau \mu(s)\, ds.
$$

Computing the derivative with respect to time τ gives

$$
\dot{\mu}(\tau) = E_t r(\tau)\, \mu(\tau) + \lambda\beta\mu(\tau) \Leftrightarrow \frac{\dot{\mu}(\tau)}{\mu(\tau)} = E_t r(\tau) + \beta\lambda. \qquad (10.5.13)
$$

Hence, the expected return is given by $E_t r(\tau) + \beta\lambda$.

The assumed independence between the interest rate and wealth in (10.5.12) is useful here but might not hold in a general equilibrium setup if wealth is a share of and the interest rate a function of the aggregate capital stock. Care should therefore be taken when using this result more generally.

- Expected growth rate

Finally, we ask what the expected growth rate and the variance of the price $v(\tau)$ is when it follows the geometric Poisson process known from (10.3.8), $dv(\tau) = \alpha v(\tau)\, d\tau + \beta v(\tau)\, dq(\tau)$.

The expected growth rate can be defined as $E_t \frac{v(\tau)-v(t)}{v(t)}$. Given that $v(t)$ is known in t, we can write this expected growth rate as $\frac{E_t v(\tau)-v(t)}{v(t)}$ where expectations are formed only with respect to the future price $v(\tau)$. Given this expression, we can follow the usual approach. The integral version of the SDE, applying the expectations operator, is

$$
E_t v(\tau) - v(t) = \alpha E_t \int_t^\tau v(s)\, ds + \beta E_t \int_t^\tau v(s)\, dq(s).
$$

Pulling the expectations operator inside the integral, using the martingale result from (10.5.7) and defining $\mu(s) \equiv E_t v(s)$ gives

$$
\mu(\tau) - v(t) = \alpha \int_t^\tau \mu(s)\, ds + \beta\lambda \int_t^\tau \mu(s)\, ds.
$$

The time-τ derivative gives $\dot{\mu}(\tau) = \alpha\mu(\tau) + \beta\lambda\mu(\tau)$ which shows that the growth rate of the mean of the price is given by $\alpha + \beta\lambda$. This is, as just discussed, also the expected growth rate of the price $v(\tau)$.

- Disentangling the mean and variance of a Poisson process

Consider the SDE $dv(t)/v(t) = \alpha dt + \kappa\beta dq(t)$ where the arrival rate of $q(t)$ is given by λ/κ. The mean of $v(\tau)$ is $E_t v(\tau) = v(t)\exp^{(\alpha+\lambda\beta)(\tau-t)}$ and thereby independent of κ. The variance, however, is given by $\text{var}_t v_1(\tau) = [E_t v(\tau)]^2 \left[\exp^{\lambda\kappa\beta^2[\tau-t]} - 1\right]$. A mean preserving spread can thus be achieved by an increase of κ : This increases the randomly occurring jump $\kappa\beta$ and reduces the arrival rate λ/κ - the mean remains unchanged, the variance increases.

- Computing the mean step-by-step for a Poisson example

Let us now consider some stochastic processes $X(t)$ described by a differential equation and ask what we know about expected values of $X(\tau)$, where τ lies in the future, i.e. $\tau > t$. We take as the first example a stochastic process similar to (10.1.8). We take as an example the specification for total factor productivity A,

$$\frac{dA(t)}{A(t)} = \alpha dt + \beta_1 dq_1(t) - \beta_2 dq_2(t), \tag{10.5.14}$$

where α and β_i are positive constants and the arrival rates of the processes are given by some constant $\lambda_i > 0$. This equation says that TFP increases in a deterministic way at the constant rate α (note that the left hand side of this differential equation gives the growth rate of $A(t)$) and jumps at random points in time. Jumps can be positive when $dq_1 = 1$ and TFP increases by the factor β_1, i.e. it increases by $\beta_1\%$, or negative when $dq_2 = 1$, i.e. TFP decreases by $\beta_2\%$.

The integral version of (10.5.14) reads (see ch. 10.1.3)

$$\begin{aligned} A(\tau) - A(t) &= \int_t^\tau \alpha A(s)\,ds + \int_t^\tau \beta_1 A(s)\,dq_1(s) - \int_t^\tau \beta_2 A(s)\,dq_2(s) \\ &= \alpha\int_t^\tau A(s)\,ds + \beta_1\int_t^\tau A(s)\,dq_1(s) - \beta_2\int_t^\tau A(s)\,dq_2(s). \end{aligned} \tag{10.5.15}$$

When we form expectations, we obtain

$$\begin{aligned} E_t A(\tau) - A(t) &= \alpha E_t\int_t^\tau A(s)\,ds + \beta_1 E_t\int_t^\tau A(s)\,dq_1(s) - \beta_2 E_t\int_t^\tau A(s)\,dq_2(s) \\ &= \alpha E_t\int_t^\tau A(s)\,ds + \beta_1\lambda_1 E_t\int_t^\tau A(s)\,ds - \beta_2\lambda_2 E_t\int_t^\tau A(s)\,ds. \end{aligned} \tag{10.5.16}$$

where the second equality used the martingale result from ch. 10.5.3, i.e. the expression in (10.5.7). Pulling the expectations operator into the integral gives

$$E_t A(\tau) - A(t) = \alpha\int_t^\tau E_t A(s)\,ds + \beta_1\lambda_1\int_t^\tau E_t A(s)\,ds - \beta_2\lambda_2\int_t^\tau E_t A(s)\,ds.$$

When we finally define $m_1(\tau) \equiv E_t A(\tau)$, we obtain a deterministic differential equation that describes the evolution of the expected value of $A(\tau)$ from the perspective of t. We first have the integral equation

$$m_1(\tau) - A(t) = \alpha \int_t^\tau m_1(s)\, ds + \beta_1 \lambda_1 \int_t^\tau m_1(s)\, ds - \beta_2 \lambda_2 \int_t^\tau m_1(s)\, ds$$

which we can then differentiate with respect to time τ by applying the rule for differentiating integrals from (4.3.3),

$$\begin{aligned} \dot{m}_1(\tau) &= \alpha m_1(\tau) + \beta_1 \lambda_1 m_1(\tau) - \beta_2 \lambda_2 m_1(\tau) \\ &= (\alpha + \beta_1 \lambda_1 - \beta_2 \lambda_2)\, m_1(\tau). \end{aligned} \tag{10.5.17}$$

We now immediately see that TFP does not increase in expected terms, more precisely $E_t A(\tau) = A(t)$, if $\alpha + \beta_1 \lambda_1 = \beta_2 \lambda_2$. Economically speaking, if the increase in TFP through the deterministic component α and the stochastic component β_1 is "destroyed" on average by the second stochastic component β_2, TFP does not increase.

10.6 Further reading and exercises

- Mathematical background

There are many textbooks on differential equations, Ito calculus, change of variable formulas and related aspects in mathematics. One that is widely used is Øksendal (1998) and some of the above material is taken from there. A more technical approach is presented in Protter (1995). Øksendal (1998, Theorem 4.1.2) covers proofs of some of the lemmas presented above. A much more general approach based on semi-martingales, and thereby covering all lemmas and CVFs presented here, is presented by Protter (1995). A classic mathematical reference is Gihman and Skorohod (1972). See also Goodman (2002) on an introduction to Brownian motion.

A special focus with a detailed formal analysis of SDEs with Poisson processes can be found in Garcia and Griego (1994). They also provide solutions of SDEs and the background for computing moments of stochastic processes. Further solutions, applied to option pricing, are provided by Das and Foresi (1996). The CVF for the combined Poisson-diffusion setup in lemma 10.2.6 is a special case of the expression in Sennewald (2007) which in turn is based on Øksendal and Sulem (2005). Øksendal and Sulem present CVFs for more general Levy processes of which the SDE (10.2.12) is a very simple special case.

The claim for the solution in (10.4.6) is an "educated guess". It builds on Arnold (1973, ch. 8.4) who provides a solution for independent Brownian motions.

- A less technical background

A very readable introduction to stochastic processes is by Ross (1993, 1996). He writes in the introduction to the first edition of his 1996 book that his text is "a nonmeasure theoretic introduction to stochastic processes". This makes this book highly accessible for economists.

An introduction with many examples from economics is Dixit and Pindyck (1994). See also Turnovsky (1997, 2000) for many applications. Brownian motion is treated extensively in Chang (2004).

The CVF for Poisson processes is most easily accessible in Sennewald (2007) or Sennewald and Wälde (2006). Sennewald (2007) provides the mathematical proofs, Sennewald and Wälde (2006) has a focus on applications. Proposition 10.2.1 is taken from Sennewald (2007) and Sennewald and Wälde (2006).

The technical background for the t_- notation is the fact that the process $x(t)$ is a so called *càdlàg* process ("continue à droite, limites à gauche"), i.e. the paths of $x(t)$ are continuous from the right with left limits. The left limit is denoted by $x(t_-) \equiv \lim_{s\uparrow t} x(s)$. See Sennewald (2007) or Sennewald and Wälde (2006) for further details and references to the mathematical literature.

- How to present technologies in continuous time

There is a tradition in economics starting with Eaton (1981) where output is represented by a stochastic differential equation as presented in ch. 10.4.1. This and similar representations of technologies are used by Epaulart and Pommeret (2003), Pommeret and Smith (2005), Turnovsky and Smith (2006), Turnovsky (2000), Chatterjee, Giuliano and Turnovsky (2004), and many others. It is well known (see e.g. footnote 4 in Grinols and Turnovsky (1998)) that this implies the possibility of negative Y. An alternative where standard Cobb-Douglas technologies are used and TFP is described by a SDE to this approach is presented in Wälde (2005) or Wälde (2011).

- Application of Poisson processes in economics

The Poisson process is widely used in finance (early references are Merton, 1969, 1971) and labour economics (in matching and search models, see e.g. Pissarides, 2000, Burdett and Mortensen, 1998 or Moscarini, 2005). See also the literature on the real options approach to investment (McDonald and Siegel, 1986, Dixit and Pindyck, 1994, Chen and Funke 2005 or Guo et al., 2005). It is also used in growth models (e.g. quality ladder models à la Aghion and Howitt, 1992 or Grossman and Helpman, 1991), in analyses of business cycles in the natural volatility tradition (e.g. Wälde, 2005), contract theory (e.g. Guriev and Kvasov, 2005), in the search approach to monetary economics (e.g. Kiyotaki and Wright, 1993 and subsequent work) and many other areas. Further examples include Toche (2005), Steger (2005), Laitner and Stolyarov (2004), Farzin et al. (1998), Hassett and Metcalf (1999), Thompson and Waldo (1994), Palokangas (2003, 2005) and Venegas-Martínez (2001).

One Poisson shock affecting many variables (as stressed in lemma 10.2.5) was used by Aghion and Howitt (1992) in their famous growth model. When deriving the budget constraint in Appendix 1 of Wälde (1999a), it is taken into consideration that a jump in the technology level affects both the capital stock directly as well as its price. Other examples include the natural volatility papers by Wälde (2002, 2005) and Posch and Wälde (2006).

"Disentangling the mean and variance of a Poisson process" is taken from Sennewald and Wälde (2006). An alternative is provided by Steger (2005) who uses two symmetric Poisson processes instead of one here. He obtains higher risk at an invariant mean by increasing the symmetric jump size.

- Other

Solutions to partial differential equations as in the option pricing example (10.4.4) are more frequently used e.g. in physics (see Black and Scholes, 1973, p. 644). Partial differential equations also appear in labour economics, however. See the Fokker-Planck equations in Bayer and Wälde (2010a,b).

Definitions and applications of martingales are provided more stringently in e.g. Øksendal (1998) or Ross (1996).

Exercises Chapter 10
Applied Intertemporal Optimization
Stochastic differential equations and rules for differentiating

1. Expectations

    (a) Assume the price of bread follows a geometric Brownian motion. What is the probability that the price will be 20% more expensive in the next year?

    (b) Consider a Poisson process with arrival rate λ. What is the probability that a jump occurs only after 3 weeks? What is the probability that 5 jumps will have occurred over the next 2 days?

2. Differentials of functions of stochastic processes I
Assume $x(t)$ and $y(t)$ are two correlated Wiener processes. Compute $d[x(t)y(t)]$, $d[x(t)/y(t)]$ and $d\ln x(t)$.

3. Differentials of functions of stochastic processes II

    (a) Show that with $F = F(x, y) = xy$ and $dx = f^x(x, y)\,dt + g^x(x, y)\,dq^x$ and $dy = f^y(x, y)\,dt + g^y(x, y)\,dq^y$, where q^x and q^y are two independent Poisson processes, $dF = xdy + ydx$.

    (b) Does this also hold for two Wiener processes q^x and q^y?

4. Correlated jump processes
Let $q_i(t)$ for $i = 1, 2, 3$ be three independent Poisson processes. Define two jump processes by $q_x(t) \equiv q_1(t) + q_2(t)$ and $q_y(t) \equiv q_1(t) + q_3(t)$. Given that $q_1(t)$ appears in both definitions, $q_x(t)$ and $q_y(t)$ are correlated jump processes. Let labour productivity in sector X and Y be driven by

$$dA(t) = \alpha dt + \beta dq_x(t),$$
$$dB(t) = \gamma dt + \delta dq_y(t).$$

Let GDP in a small open economy (with internationally given constant prices p_x and p_y) be given by

$$Y(t) \equiv p_x A(t) L_x + p_y B(t) L_y.$$

(a) Given an economic interpretation to equations $dA(t)$ and $dB(t)$, based on $q_i(t)$, $i = 1, 2, 3$.

(b) How does GDP evolve over time in this economy if labour allocation L_x and L_y is invariant? Express the differential $dY(t)$ by using dq_x and dq_y if possible.

5. Deriving a budget constraint
Consider a household who owns some wealth $a = n_1 v_1 + n_2 v_2$, where n_i denotes the number of shares held by the household and v_i is the value of the share. Assume that the value of the share evolves according to

$$dv_i = \alpha_i v_i dt + \beta_i v_i dq_i.$$

Assume further that the number of shares held by the household changes according to

$$dn_i = \frac{\chi_i (\pi_1 n_1 + \pi_2 n_2 + w - e)}{v_i} dt,$$

where χ_i is the share of savings used for buying stock i.

(a) Give an interpretation (in words) of the last equation.

(b) Derive the household's budget constraint.

6. Option pricing
Assume the price of an asset follows $dS/S = \alpha dt + \sigma dz + \beta dq$ (as in Merton, 1976), where z is Brownian motion and q is a Poisson process. This is a generalization of (10.3.1) where $\beta = 0$. How does the differential equation look like that determines the price of an option on this asset?

7. Martingales

(a) The weather tomorrow will be just the same as today. Is this a martingale?

(b) Let $z(s)$ be Brownian motion. Show that $Y(t)$ defined by

$$Y(t) \equiv \exp\left[-\int_0^t f(s)\, dz(s) - \frac{1}{2}\int_0^t f^2(s)\, ds\right] \tag{10.6.1}$$

is a martingale.

(c) Show that $X(t)$ defined by

$$X(t) \equiv \exp\left[\int_0^t f(s)\, dz(s) - \frac{1}{2}\int_0^t f^2(s)\, ds\right]$$

is also a martingale.

8. Expectations - Poisson
   Assume that you are in t and form expectations about future arrivals of the Poisson process $q(t)$. Prove the following statement by using lemma 10.5.1: The number of expected arrivals in the time interval $[\tau, s]$ equals the number of expected arrivals in a time interval of the length $\tau - s$ for any $s > t$,

$$E_t(c^{k[q(\tau)-q(s)]}) = E(c^{k[q(\tau)-q(s)]}) = e^{(c^k-1)\lambda(\tau-s)}, \quad \tau > s > t, \quad c, k = \text{const.}$$

   Hint: If you want to cheat, look at the appendix to Posch and Wälde (2006). It is available for example on www.waelde.com/publications.

9. Expected values
   Show that the growth rate of the mean of $x(t)$ described by the geometric Brownian motion

$$dx(t) = ax(t)\,dt + bx(t)\,dz(t)$$

   is given by a.

   (a) Do so by using the integral version of this SDE and compute the increase of the expected value of $x(t)$.
   (b) Do the same as in (a) but solve first the SDE and compute expectations by using this solution.
   (c) Compute the covariance of $x(\tau)$ and $x(s)$ for $\tau > s \geq t$.
   (d) Compute the density function $f(x(\bar{\tau}))$ for one specific point in time $\bar{\tau} > t$. Hint: Compute the variance of $x(\bar{\tau})$ and the expected value of $x(\bar{\tau})$ as in (a) to (c). What type of distribution does $x(\tau)$ come from? Compute the parameters μ and σ^2 of this distribution for one specific point in time $\tau = \bar{\tau}$.

10. Expected returns
    Consider the budget constraint

$$da(t) = \{ra(t) + w - c(t)\}\,dt + \beta a(t)\,dz(t).$$

    (a) What is the expected return for wealth? Why does this expression differ from (10.5.13)?
    (b) What is the variance of wealth?

11. "Differential-representation" of a technology

    (a) Consider the "differential-representation" of a technology, $dY(t) = AKdt + \sigma K dz(t)$, as presented in (10.4.1). Compute expected output of $Y(\tau)$ for $\tau > t$ and the variance of $Y(\tau)$.

(b) Consider a more standard representation by $Y(t) = A(t)K$ and let TFP follow $dA(t)/A(t) = gdt + \sigma dz(t)$ as in (10.4.2). What is the expected output level $E_t Y(\tau)$ and what is its variance?

12. Solving stochastic differential equations
Consider the differential equation

$$dx(t) = [a(t) - x(t)]\,dt + c_1(t)\,x(t)\,dz_1(t) + g_2(t)\,dz_2(t)$$

where z_1 and z_2 are two correlated Brownian motions.

(a) What is the solution of this differential equation?

(b) Use Ito's Lemma to show that your solution is a solution.

13. Dynamic and intertemporal budget constraints - Brownian motion
Consider the dynamic budget constraint

$$da(\tau) = (r(\tau)\,a(\tau) + w(\tau) - p(\tau)\,c(\tau))\,dt + \sigma a(\tau)\,dz(\tau),$$

where $z(\tau)$ is Brownian motion.

(a) Show that the intertemporal version of this budget constraint, using a no-Ponzi game condition, can be written as

$$\int_t^\infty e^{-\varphi(\tau)} p(\tau)\,c(\tau)\,d\tau = a_t + \int_t^\infty e^{-\varphi(\tau)} w(\tau)\,d\tau \tag{10.6.2}$$

where the discount factor $\varphi(\tau)$ is given by

$$\varphi(\tau) = \int_t^\tau \left(r(s) - \frac{1}{2}\sigma^2\right) ds + \int_t^\tau \sigma dz(\tau).$$

(b) Now assume we are willing to assume that intertemporal budget constraints need to hold in expectations only and not in realizations: we require only that agents balance at each instant *expected* consumption expenditure to current financial wealth plus *expected* labour income. Show that the intertemporal budget constraint then simplifies to

$$E_t \int_t^\infty e^{-\int_t^\tau \left(r(s) - \sigma^2\right) ds} p(\tau)\,c(\tau)\,d\tau = a_t + E_t \int_t^\infty e^{-\int_t^\tau \left(r(s) - \sigma^2\right) ds} w(\tau)\,d\tau$$

i.e. all stochastic terms drop out of the discount factor but the variance stays there.

Chapter 11

Infinite horizon models

We now return to our main concern: How to solve maximization problems. We first look at optimal behaviour under Poisson uncertainty where we analyse cases for uncertain asset prices and labour income. We then switch to Brownian motion and look at capital asset pricing as an application.

11.1 Intertemporal utility maximization under Poisson uncertainty

11.1.1 The setup

Let us consider an individual that tries to maximize his objective function that is given by

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau))\, d\tau. \tag{11.1.1}$$

The structure of this objective function is identical to the one we know from deterministic continuous time models in e.g. (5.1.1) or (5.6.1): We are in t today, the time preference rate is $\rho > 0$, instantaneous utility is given by $u(c(\tau))$. Given the uncertain environment the individual lives in, we now need to form expectations as consumption in future points τ in time is unknown.

When formulating the budget constraint of a household, we have now seen at various occasions that it is a good idea to derive it from the definition of wealth of a household. We did so in discrete time models in ch. 2.5.5 and in continuous time models in ch. 4.4.2. Deriving the budget constraint in stochastic continuous time models is especially important as a budget constraint in an economy where the fundamental source of uncertainty is Brownian motion looks very different from one where uncertainty stems from Poisson or Levy processes. For this first example, we use the budget constraint (10.3.9) derived in ch. 10.3.2,

$$da(t) = \{r(t)\, a(t) + w(t) - pc(t)\}\, dt + \beta a(t)\, dq(t), \tag{11.1.2}$$

where the interest rate $r(t) \equiv \alpha + \pi(t)/v(t)$ was defined as the deterministic rate of change α of the price of the asset (compare the equation for the evolution of assets in (10.3.8)) plus dividend payments $\pi(t)/v(t)$. We treat the price p here as a constant (see the exercises for an extension). Following the tilde notation from (10.1.7), we can express wealth $\tilde{a}(t)$ after a jump by

$$\tilde{a}(t) = (1+\beta)\, a(t). \tag{11.1.3}$$

The budget constraint of this individual reflects standard economic ideas about budget constraints under uncertainty. As visible in the derivation in ch. 10.3.2, the uncertainty for this household stems from uncertainty about the evolution of the price (reflected in β) of the asset he saves in. No statement was made about the evolution of the wage $w(t)$. Hence, we take $w(t)$ here as parametric, i.e. if there are stochastic changes, they all come as a surprise and are therefore not anticipated. The household does take into consideration, however, the uncertainty resulting from the evolution of the price $v(t)$ of the asset. In addition to the deterministic growth rate α of $v(t)$, $v(t)$ changes in a stochastic way by jumping occasionally by β percent (again, see (10.3.8)). The returns to wealth $a(t)$ are therefore uncertain and are composed of the "usual" $r(t)\, a(t)$ term and the stochastic $\beta a(t)$ term.

11.1.2 Solving by dynamic programming

We will solve this problem as before by dynamic programming methods. Note, however, that it is not obvious whether the above problem can be solved by dynamic programming methods. In principle, a proof is required that dynamic programming indeed yields necessary and sufficient conditions for an optimum. While proofs exist for bounded instantaneous utility function $u(t)$, such a proof did not exist until recently for unbounded utility functions. Sennewald (2007) extends the standard proofs and shows that dynamic programming can also be used for unbounded utility functions. We can therefore follow the usual three-step approach to dynamic programming here as well.

- DP1: Bellman equation and first-order conditions

The first tool we need to derive rules for optimal behavior is the Bellman equation. Defining the optimal program as $V(a) \equiv \max_{\{c(\tau)\}} U(t)$ subject to the constraint (11.1.2), this equation is given by (see Sennewald and Wälde, 2006, or Sennewald, 2007)

$$\rho V(a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t)) \right\}. \tag{11.1.4}$$

The Bellman equation has this basic form for "most" maximization problems in continuous time. It can therefore be taken as the starting point for other maximization problems as well, independently, for example, of whether uncertainty is driven by Poisson processes, Brownian motion or Levy processes. We will see examples of related problems later (see ch. 11.1.4, ch. 11.2.2 or ch. 11.3.2) and discuss then how to adjust certain features of this "general" Bellman equation. In this equation, the variable $a(t)$ represents the state variable, in our case wealth of the individual. See ch. 6.1 on dynamic programming in a deterministic continuous time setup for a detailed intuitive discussion of the structure of the Bellman equation. The discussion on "what is a state variable" of ch. 3.4.2 applies here as well.

Given the general form of the Bellman equation in (11.1.4), we need to compute the differential $dV(a(t))$. Given the evolution of $a(t)$ in (11.1.2) and the CVF from (10.2.8), we find

$$dV(a) = V'(a)\{ra + w - pc\}dt + \{V(a + \beta a) - V(a)\}dq.$$

In contrast to the CVF notation in for example (10.2.8), we use here and in what follows simple derivative signs like $V'(a)$ as often as possible in contrast to for example $V_a(a)$. This is possible as long as the functions, like the value function $V(a)$ here, have one argument only. Forming expectations about $dV(a(t))$ is easy and they are given by

$$E_t dV(a(t)) = V'(a)\{ra + w - pc\}dt + \{V(\tilde{a}) - V(a)\}E_t dq.$$

The first term, the "dt-term" is known in t: The current state $a(t)$ and all prices are known and the shadow price $V'(a)$ is therefore also known. As a consequence, expectations need to be applied only to the "dq-term". The first part of the "dq-term", the expression

$V((1+\beta)a) - V(a)$ is also known in t as again $a(t)$, parameters and the function V are all non-stochastic. We therefore only have to compute expectations about dq. From (10.5.3), we know that $E_t[q(\tau) - q(t)] = \lambda[\tau - t]$. Now replace $q(\tau) - q(t)$ by dq and $\tau - t$ by dt and find $E_t dq = \lambda dt$. The Bellman equation therefore reads

$$\rho V(a) = \max_{c(t)} \{u(c(t)) + V'(a)[ra + w - pc] + \lambda [V((1+\beta)a) - V(a)]\}. \quad (11.1.5)$$

Note that forming expectations the way just used is, say, informal. Doing it in the more stringent way introduced in ch. 10.5.4 would, however, lead to identical results.

The first-order condition is

$$u'(c) = V'(a)p. \quad (11.1.6)$$

As always, (current) utility from an additional unit of consumption $u'(c)$ must equal (future) utility from an additional unit of wealth $V'(a)$, multiplied by the price p of the consumption good, i.e. by the number of units of wealth for which one can buy one unit of the consumption good.

- DP2: Evolution of the costate variable

In order to understand the evolution of the marginal value $V'(a)$ of the optimal program, i.e. the evolution of the costate variable, we need to (i) compute the partial derivative of the maximized Bellman equation with respect to assets and (ii) compute the differential $dV'(a)$ by using the CVF and insert the partial derivative into this expression. These two steps correspond to the two steps in DP2 in the deterministic continuous time setup of ch. 6.1.

(i) In the first step, we state the maximized Bellman equation from (11.1.5) as the Bellman equation where controls are replaced by their optimal values,

$$\rho V(a) = u(c(a)) + V'(a)[ra + w - pc(a)] + \lambda [V(\tilde{a}) - V(a)].$$

We then compute again the derivative with respect to the state - as in discrete time and deterministic continuous time setups - as this gives us an expression for the shadow price $V'(a)$. In contrast to the previous emphasis on Ito's Lemmas and CVFs, we can use for this step standard rules from algebra as we compute the derivative for a given state a - the state variable is held constant and we want to understand the derivative of the function $V(a)$ with respect to a. We do not compute the differential of $V(a)$ and ask how the value function changes as a function of a change in a. Therefore, using the envelope theorem,

$$\rho V'(a) = V'(a)r + V''(a)[ra + w - pc] + \lambda [V'(\tilde{a})[1+\beta] - V'(a)]. \quad (11.1.7)$$

We used here the definition of $\tilde{a}$ given in (11.1.3).

(ii) In the second step, the differential of the shadow price $V'(a)$ is computed. Here, we do need a change of variable formula. Hence, given the evolution of $a(t)$ in (11.1.2),

$$dV'(a) = V''(a)[ra + w - pc]dt + [V'(\tilde{a}) - V'(a)]dq. \quad (11.1.8)$$

Finally, replacing $V''(a)[ra + w - pc]$ in (11.1.8) by the same expression from (11.1.7) gives

$$dV'(a) = \{(\rho - r) V'(a) - \lambda [V'(\tilde{a})[1+\beta] - V'(a)]\} dt + \{V'(\tilde{a}) - V'(a)\} dq.$$

- DP3: Inserting first-order conditions

Finally, we can replace the marginal value by marginal utility from the first-order condition (11.1.6). In this step, we employ that p is constant and therefore $dV'(a) = p^{-1} du'(c)$. Hence, the rule describing the evolution of marginal utility reads

$$du'(c) = \{(\rho - r) u'(c) - \lambda [u'(\tilde{c})[1+\beta] - u'(c)]\} dt + \{u'(\tilde{c}) - u'(c)\} dq. \tag{11.1.9}$$

Note that the constant price p dropped out. This rule shows how marginal utility changes in a deterministic and stochastic way.

11.1.3 The Keynes-Ramsey rule

The dynamic programming approach provided us with an expression in (11.1.9) which describes the evolution of marginal utility from consumption. While there is a one-to-one mapping from marginal utility to consumption which would allow some inferences about consumption from (11.1.9), it would nevertheless be more useful to have a Keynes-Ramsey rule for optimal consumption itself.

- The evolution of consumption

If we want to know more about the evolution of consumption, we can use the CVF formula as follows. Let $f(.)$ be the inverse function for u', i.e. $f(u'(c)) = c$, and apply the CVF to $f(u'(c))$. This gives

$$\begin{aligned} df(u'(c)) &= f'(u'(c)) \{(\rho - r) u'(c) - \lambda [u'(\tilde{c})[1+\beta] - u'(c)]\} dt \\ &\quad + \{f(u'(\tilde{c})) - f(u'(c))\} dq. \end{aligned}$$

As $f(u'(c)) = c$, we know that $f(u'(\tilde{c})) = \tilde{c}$ and $f'(u'(\tilde{c})) \equiv \frac{df(u'(c))}{du'(c)} = \frac{dc}{du'(c)} = \frac{1}{u''(c)}$. Hence,

$$\begin{aligned} dc &= \frac{1}{u''(c)} \{(\rho - r) u'(c) - \lambda [u'(\tilde{c})[1+\beta] - u'(c)]\} dt + \{\tilde{c} - c\} dq \Leftrightarrow \\ -\frac{u''(c)}{u'(c)} dc &= \left\{ r - \rho + \lambda \left[\frac{u'(\tilde{c})}{u'(c)} [1+\beta] - 1 \right] \right\} dt - \frac{u''(c)}{u'(c)} \{\tilde{c} - c\} dq. \end{aligned} \tag{11.1.10}$$

This is the Keynes-Ramsey rule that describes the evolution of consumption under optimal behaviour for a household that faces interest rate uncertainty resulting from Poisson processes. This equation is useful to understand, for example, economic fluctuations in a natural volatility setup. It corresponds to its deterministic pendant in (5.6.4) in ch. 5.6.1: By setting $\lambda = 0$ here (implying $dq = 0$), noting that we treated the price as a constant and dividing by dt, we obtain (5.6.4).

- A specific utility function

Let us now assume that the instantaneous utility function is given by the widely used constant relative risk aversion (CRRA) utility function,

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0. \tag{11.1.11}$$

Then, the Keynes-Ramsey rule becomes

$$\sigma \frac{dc}{c} = \left\{ r - \rho + \lambda \left[(1+\beta) \left(\frac{c}{\tilde{c}} \right)^{\sigma} - 1 \right] \right\} dt + \sigma \left\{ \frac{\tilde{c}}{c} - 1 \right\} dq. \tag{11.1.12}$$

The left-hand side gives the proportional change of consumption times σ, the inverse of the intertemporal elasticity of substitution σ^{-1}. This corresponds to $\sigma \dot{c}/c$ in the deterministic Keynes-Ramsey rule in e.g. (5.6.5). Growth of consumption depends on the right-hand side in a deterministic way on the usual difference between the interest rate and time preference rate plus the "$\lambda-$term" which captures the impact of uncertainty. When we want to understand the meaning of this term, we need to find out whether consumption jumps up or down, following a jump of the Poisson process. When β is positive, the household holds more wealth and consumption increases. Hence, the ratio $c/\tilde{c}$ is smaller than unity and the sign of the bracket term $(1+\beta)\left(\frac{c}{\tilde{c}}\right)^{\sigma} - 1$ is qualitatively unclear. If it is positive, consumption growth is faster in a world where wealth occasionally jumps by β percent.

The dq-term gives discrete changes in the case of a jump in q. It is, however, tautological: When q jumps and $dq = 1$ and $dt = 0$ for this small instant of the jump, (11.1.12) says that $\sigma dc/c$ on the left-hand side is given by $\sigma \{\tilde{c}/c - 1\}$ on the right hand side. As the left hand side is by definition of dc given by $\sigma [\tilde{c} - c]/c$, both sides are identical. Hence, the level of $\tilde{c}$ after the jump needs to be determined in an alternative way.

11.1.4 Optimal consumption and portfolio choice

This section analyses a more complex maximization problem than the one presented in ch. 11.1.1. In addition to the consumption-saving trade-off, it includes a portfolio choice problem. Interestingly, the solution is much simpler to work with as $\tilde{c}$ can explicitly be computed and closed-form solutions can easily be found.

- The maximization problem

Consider a household that is endowed with some initial wealth $a_0 > 0$. At each instant, the household can invest its wealth $a(t)$ in both a risky and a safe asset. The share of wealth the household holds in the risky asset is denoted by $\theta(t)$. The price $v_1(t)$ of one unit of the risky asset obeys the SDE

$$dv_1(t) = r_1 v_1(t)\, dt + \beta v_1(t)\, dq(t), \tag{11.1.13}$$

where $r_1 \in \mathbb{R}$ and $\beta > 0$. That is, the price of the risky asset grows at each instant with a fixed rate r_1 and at random points in time it jumps by β percent. The randomness comes from the well-known Poisson process $q(t)$ with arrival rate λ. The price $v_2(t)$ of one unit of the safe asset is assumed to follow

$$dv_2(t) = r_2 v_2(t)\,dt, \tag{11.1.14}$$

where $r_2 \geq 0$. Let the household receive a fixed wage income w and spend $c(t) \geq 0$ on consumption. Then, in analogy to ch. 10.3.2, the household's budget constraint reads

$$da(t) = \{[\theta(t) r_1 + (1 - \theta(t)) r_2] a(t) + w - c(t)\} dt + \beta\theta(t) a(t) dq(t). \tag{11.1.15}$$

We allow wealth to become negative but we could assume that debt is always covered by the household's lifetime labour income discounted with the safe interest rate r_2, i.e. $a(t) > -w/r_2$.

Let intertemporal preferences of households be identical to the previous maximization problem - see (11.1.1). The instantaneous utility function is again characterized by CRRA as in (11.1.11), $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$. The control variables of the household are the nonnegative consumption stream $\{c(t)\}$ and the share $\{\theta(t)\}$ held in the risky asset. To avoid a trivial investment problem, we assume

$$r_1 < r_2 < r_1 + \lambda\beta. \tag{11.1.16}$$

That is, the guaranteed return of the risky asset, r_1, is lower than the return of the riskless asset, r_2, whereas, on the other hand, the expected return of the risky asset, $r_1 + \lambda\beta$, shall be greater than r_2. If r_1 was larger than r_2, the risky asset would dominate the riskless one and no one would want to hold positive amounts of the riskless asset. If r_2 exceeded $r_1 + \lambda\beta$, the riskless asset would dominate.

- DP1: Bellman equation and first-order conditions

Again, the first step of the solution of this maximization problem requires a Bellman equation. Define the value function V again as $V(a(t)) \equiv \max_{\{c(\tau),\theta(\tau)\}} U(t)$. The basic Bellman equation is taken from (11.1.4). When computing the differential $dV(a(t))$ and taking into account that there are now two control variables, the Bellman equation reads

$$\rho V(a) = \max_{c(t),\theta(t)} \{u(c) + [(\theta r_1 + (1-\theta) r_2) a + w - c] V'(a) + \lambda [V(\tilde{a}) - V(a)]\}, \tag{11.1.17}$$

where $\tilde{a} \equiv (1 + \theta\beta) a$ denotes the post-jump wealth if at wealth a a jump in the risky asset price occurs. The first-order conditions which any optimal path must satisfy are given by

$$u'(c) = V'(a) \tag{11.1.18}$$

and

$$V'(a)(r_1 - r_2) a + \lambda V'(\tilde{a}) \beta a = 0. \tag{11.1.19}$$

While the first first-order condition equates as always marginal utility with the shadow price, the second first-order condition determines optimal investment of wealth into assets 1 and 2, i.e. the optimal share θ. The latter first-order condition contains a deterministic and a stochastic term and households hold their optimal share if these two components just add up to zero. Assume, consistent with (11.1.16), that $r_1 < r_2$. If we were in a deterministic world, i.e. $\lambda = 0$, households would then only hold asset 2 as its return is higher. In a stochastic world, the lower instantaneous return on asset 1 is compensated by the fact that, as (11.1.13) shows, the price of this asset jumps up occasionally by the percentage β. Lower instantaneous returns r_1 paid at each instant are therefore compensated for by large occasional positive jumps.

As this first-order condition also shows, returns and jumps per se do not matter: The difference $r_1 - r_2$ in returns is multiplied by the shadow price $V'(a)$ of capital and the effect of the jump size times its frequency, $\lambda\beta$, is multiplied by the shadow price $V'(\tilde{a})$ of capital after a jump. What matters for the household decision is therefore the impact of holding wealth in one or the other asset on the overall value from behaving optimally, i.e. on the value function $V(a)$. The channels through which asset returns affect the value function is first the impact on wealth and second the impact of wealth on the value function i.e. the shadow price of wealth.

We can now immediately see why this more complex maximization problem yields simpler solutions: Replacing in equation (11.1.19) $V'(a)$ with $u'(c)$ according to (11.1.18) yields for $a \neq 0$

$$\frac{u'(\tilde{c})}{u'(c)} = \frac{r_2 - r_1}{\lambda\beta}, \tag{11.1.20}$$

where $\tilde{c}$ denotes the optimal consumption choice for $\tilde{a}$. Hence, the ratio for optimal consumption after and before a jump is constant. If we assume, for example, a CRRA utility function as in (11.1.11), this jump is given by

$$\frac{\tilde{c}}{c} = \left(\frac{\lambda\beta}{r_2 - r_1}\right)^{1/\sigma}. \tag{11.1.21}$$

No such result on relative consumption before and after the jump is available for the maximization problem without a choice between a risky and a riskless asset.

Since by assumption (11.1.16) the term on the right-hand side is greater than 1, this equation shows that consumption jumps upwards if a jump in the risky asset price occurs. This result is not surprising, as, if the risky asset price jumps upwards, so does the household's wealth.

- DP2: Evolution of the costate variable

In the next step, we compute the evolution of $V'(a(t))$, the shadow price of wealth. Assume that V is twice continuously differentiable. Then, due to budget constraint (11.1.15), the CVF from (10.2.8) yields

$$\begin{aligned} dV'(a) = &\{[\theta r_1 + (1-\theta) r_2] a + w - c\} V''(a)\, dt \\ &+ \{V'(\tilde{a}) - V'(a)\} \, dq(t). \end{aligned} \tag{11.1.22}$$

Differentiating the maximized Bellman equation yields under application of the envelope theorem

$$\rho V'(a) = \{[\theta r_1 + (1-\theta) r_2] a + w - c\} V''(a) \\ + \{\theta r_1 + [1-\theta] r_2\} V'(a) + \lambda \{V'(\tilde{a}) [1+\theta\beta] - V'(a)\}.$$

Rearranging gives

$$\{[\theta r_1 + (1-\theta) r_2] a + w - c\} V''(a) \\ = \{\rho - [\theta r_1 + (1-\theta) r_2]\} V'(a) - \lambda \{V'(\tilde{a}) [1+\theta\beta] - V'(a)\}.$$

Inserting this into (11.1.22) yields

$$dV'(a) = \left\{ \begin{array}{c} \{\rho - [\theta r_1 + (1-\theta) r_2]\} V'(a) \\ -\lambda \{[1+\theta\beta] V'(\tilde{a}) - V'(a)\} \end{array} \right\} dt + \{V'(\tilde{a}) - V'(a)\} dq(t).$$

- DP3: Inserting first-order conditions

Replacing $V'(a)$ by $u'(c)$ following the first-order condition (11.1.18) for optimal consumption, we obtain

$$du'(c) = \left\{ \begin{array}{c} \{\rho - [\theta r_1 + (1-\theta) r_2]\} u'(c) \\ -\lambda \{[1+\theta\beta] u'(\tilde{c}) - u'(c)\} \end{array} \right\} dt + \{u'(\tilde{c}) - u'(c)\} dq(t).$$

Now applying the CVF again to $f(x) = (u')^{-1}(x)$ and using (11.1.20) leads to the Keynes-Ramsey rule for general utility functions u,

$$-\frac{u''(c)}{u'(c)} dc = \left\{ \theta r_1 + [1-\theta] r_2 - \rho + \lambda \left[\frac{r_2 - r_1}{\lambda\beta} [1+\theta\beta] - 1 \right] \right\} dt - \frac{u''(c)}{u'(c)} \{\tilde{c} - c\} dq(t).$$

As $\tilde{c}$ is also implicitly determined by (11.1.20), this Keynes-Ramsey rule describes the evolution of consumption under Poisson uncertainty without the $\tilde{c}$ term. This is the crucial modelling advantage of introducing an additional asset into a standard consumption-saving problem. Apart from this simplification, the structure of this Keynes-Ramsey rule is identical to the one in (11.1.10) without a second asset.

- A specific utility function

For the CRRA utility function as in (11.1.11), the elimination of $\tilde{c}$ becomes even simpler and we obtain with (11.1.21)

$$\frac{dc(t)}{c(t)} = \frac{1}{\sigma} \left\{ r_2 - \lambda \left[1 - \frac{r_2 - r_1}{\lambda\beta} \right] - \rho \right\} dt + \left\{ \left(\frac{\lambda\beta}{r_2 - r_1} \right)^{1/\sigma} - 1 \right\} dq(t).$$

The optimal change in consumption can thus be expressed in terms of well-known parameters. As long as the price of the risky asset does not jump, optimal consumption grows constantly by the rate $\left[r_2 - \lambda\left(1 - \frac{r_2 - r_1}{\lambda\beta}\right) - \rho\right]/\sigma$. The higher the risk-free interest rate, r_2, and the lower the guaranteed interest rate of the risky asset, r_1, the discrete growth rate, β, the probability of a price jump, λ, the time preference rate, ρ, and the risk aversion parameter, σ, the higher becomes the consumption growth rate. If the risky asset price jumps, consumption jumps as well to its new higher level $c(t) = [(\lambda\beta)/(r_2 - r_1)]^{1/\sigma} c(t)$. Here the growth rate depends positively on λ, β, and r_1, whereas r_2 and σ have a negative influence.

11.1.5 Other ways to determine $\tilde{c}$

The question of how to determine $\tilde{c}$ without an additional asset is in principle identical to determining the initial level of consumption, given a deterministic Keynes-Ramsey rule as in for example (5.1.6). Whenever a jump in c following (11.1.12) occurs, the household faces the issue of how to choose the initial level of consumption after the jump. In principle, the level $\tilde{c}$ is therefore pinned down by some transversality condition. In practice, the literature offers two ways, as to how $\tilde{c}$ can be determined.

- One asset and idiosyncratic risk

When households determine optimal savings only, as in our setup where the only first-order condition is (11.1.6), $\tilde{c}$ can be determined (in principle) if we assume that the value function is a function of wealth only - which would be the case in our household example if the interest rate and wages did not depend on q. This would naturally be the case in idiosyncratic risk models where aggregate variables do not depend on individual uncertainty resulting from q. The first-order condition (11.1.6) then reads $u'(c) = V'(a)$ (with $p = 1$). This is equivalent to saying that consumption is a function of the only state variable, i.e. wealth a, $c = c(a)$. An example for $c(a)$ is plotted in the following figure.

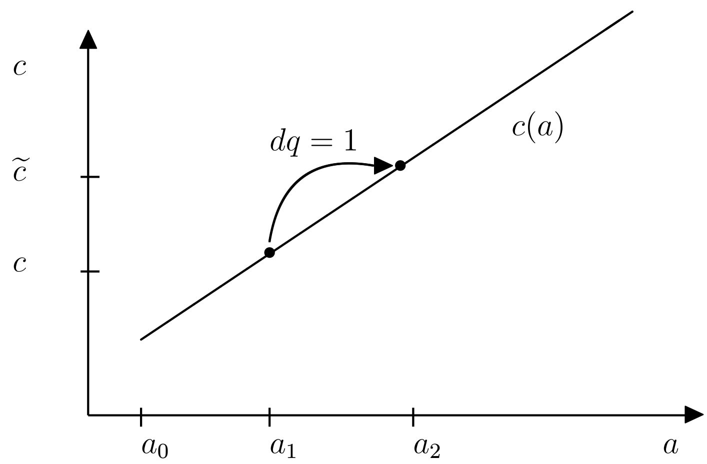


Figure 11.1.1 *Consumption c as a function of wealth a*

As consumption c does not depend on q directly, we must be at the same consumption level $c(a)$, no matter "how we got there", i.e. no matter how many jumps took place before. Hence, if we jump from some a_1 to a_2 because $dq = 1$, we are at the same consumption level $c(a_2)$ as if we had reached a_2 smoothly without jump. The consumption level $\tilde{c}$ after a jump is therefore the consumption level that belongs to the asset level after the jump according to the policy function $c(a)$ plotted in the above figure, $\tilde{c} = c(a_2)$. See Schlegel (2004) for a numerical implementation of this approach.

- Finding value functions

A very long tradition exists in economics where value functions are found by an "educated guess". Experience tells us - based on first examples by Merton (1969, 1971) - what value functions generally look like. It is then possible to find, after some attempts, the value function for some specific problem. This then implies explicit - so called closed-form solutions - for consumption (or any other control variable). For the saving and investment problem in ch. 11.1.4, Sennewald and Wälde (2006, sect. 3.4) presented a value function and closed-form solutions for $a \neq 0$ of the form

$$V(a) = \frac{\psi^{-\sigma}\left[a + \frac{w}{r_2}\right]^{1-\sigma} - \rho^{-1}}{1-\sigma}, \quad c = \psi\left[a + \frac{w}{r_2}\right], \quad \theta = \left[\left(\frac{\lambda\beta}{r_2 - r_1}\right)^{\frac{1}{\sigma}} - 1\right]\frac{a + \frac{w}{r_2}}{\beta a}.$$

Consumption is a constant share (ψ is a collection of parameters) out of the total wealth, i.e. financial wealth a plus "human wealth" w/r_2 (the present value of current and future labour income). The optimal share θ depends on total wealth as well, but also on interest rates, the degree of risk-aversion and the level β of the jump of the risky price in (11.1.13). Hence, it is possible to work with complex stochastic models that allow to analyse many interesting real-world features and nevertheless end up with explicit closed-form solutions. Many further examples exist - see ch. 11.6 on "further reading".

- Finding value functions for special cases

As we have just seen, value functions and closed-form solutions can be found for some models which have "nice features". For a much larger class of models - which are then standard models - closed-form solutions cannot be found for general parameter sets. Economists then either go for numerical solutions, which preserves a certain generality as in principle the properties of the model can be analyzed for all parameter values, or they restrict the parameter set in a useful way. Useful means that with some parameter restriction, value functions can be found again and closed-form solutions are again possible.

11.1.6 Expected growth

Let us now try to understand the impact of uncertainty on expected growth. In order to compute expected growth of consumption from realized growth rates (11.1.12), we rewrite this equation as

$$\sigma dc(t) = \left\{ r(t) - \rho + \lambda \left[(1+\beta) \left(\frac{c(t)}{\tilde{c}(t)} \right)^{\sigma} - 1 \right] \right\} c(t)\, dt + \sigma \{ \tilde{c}(t) - c(t) \} dq.$$

Expressing it in its integral version as in (10.5.15), we obtain for $\tau > t$,

$$\sigma [c(\tau) - c(t)] = \int_t^\tau \left\{ r(s) - \rho + \lambda \left[(1+\beta) \left(\frac{c(s)}{\tilde{c}(s)} \right)^{\sigma} - 1 \right] \right\} c(s)\, ds \\ + \sigma \int_t^\tau \{ \tilde{c}(s) - c(s) \} dq(s).$$

Applying the expectations operator, given knowledge in t, yields

$$E_t c(\tau) - c(t) = \frac{1}{\sigma} E_t \int_t^\tau \left\{ r(s) - \rho + \lambda \left[(1+\beta) \left(\frac{c(s)}{\tilde{c}(s)} \right)^{\sigma} - 1 \right] \right\} c(s)\, ds \\ + E_t \int_t^\tau \{ \tilde{c}(s) - c(s) \} dq(s).$$

Using again the martingale result from ch. 10.5.3 as already in (10.5.16), i.e. the expression in (10.5.7), we replace $E_t \int_t^\tau \{ \tilde{c}(s) - c(s) \} dq(s)$ by $\lambda E_t \int_t^\tau \{ \tilde{c}(s) - c(s) \} ds$, i.e.

$$E_t c(\tau) - c(t) = \frac{1}{\sigma} E_t \int_t^\tau \left\{ r(s) - \rho + \lambda \left[(1+\beta) \left(\frac{c(s)}{\tilde{c}(s)} \right)^{\sigma} - 1 \right] \right\} c(s)\, ds \\ + \lambda E_t \int_t^\tau \{ \tilde{c}(s) - c(s) \} ds.$$

Differentiating with respect to time τ yields

$$dE_t c(\tau)/d\tau = \frac{1}{\sigma} E_t \left\{ \left\{ r(\tau) - \rho + \lambda \left[(1+\beta) \left(\frac{c(\tau)}{\tilde{c}(\tau)} \right)^{\sigma} - 1 \right] \right\} c(\tau) \right\} + \lambda E_t \{ \tilde{c}(\tau) - c(\tau) \}.$$

Let us now put into the perspective of time τ, i.e. let's move time from t to τ and let's ask what expected growth of consumption is. This shift in time means formally that our expectations operator becomes E_τ and we obtain

$$\frac{dE_\tau c(\tau)/d\tau}{c(\tau)} = \frac{1}{\sigma} E_\tau \left\{ r(\tau) - \rho + \lambda \left[(1+\beta) \left(\frac{c(\tau)}{\tilde{c}(\tau)} \right)^{\sigma} - 1 \right] \right\} + \lambda E_\tau \left\{ \frac{\tilde{c}(\tau)}{c(\tau)} - 1 \right\}.$$

In this step we used the fact that due to this shift in time, $c(\tau)$ is now known and we can pull it out of the expectations operator and divide by it. Combining brackets yields

$$\frac{dE_\tau c(\tau)/d\tau}{c(\tau)} = \frac{1}{\sigma} E_\tau \left\{ r(\tau) - \rho + \lambda \left[(1+\beta) \left(\frac{c(\tau)}{\tilde{c}(\tau)} \right)^{\sigma} - 1 \right] + \sigma\lambda \left[\frac{\tilde{c}(\tau)}{c(\tau)} - 1 \right] \right\} \\ = \frac{1}{\sigma} E_\tau \left\{ r(\tau) - \rho + \lambda \left[(1+\beta) \left(\frac{c(\tau)}{\tilde{c}(\tau)} \right)^{\sigma} + \sigma \frac{\tilde{c}(\tau)}{c(\tau)} - 1 - \sigma \right] \right\}.$$

11.2 Matching on the labour market: where value functions come from

Value functions are widely used in matching models. Examples are unemployment with frictions models of the Mortensen-Pissarides type or shirking models of unemployment a la Shapiro and Stiglitz (1984). These value functions can be understood very easily on an intuitive level, but they really come from a maximization problem of households. In order to understand when value functions as the ones used in the just-mentioned examples can be used (e.g. under the assumption of no saving, or being in a steady state), we now derive value functions in a general way and then derive special cases used in the literature.

11.2.1 A household

Let wealth a of a household evolve according to

$$da = \{ra + z - c\}\, dt.$$

Wealth increases per unit of time dt by the amount da which depends on current savings $ra + z - c$. Labour income is denoted by z which includes income w when employed and unemployment benefits b when unemployed, $z = w, b$. Labour income follows a stochastic Poisson differential equation as there is job creation and job destruction. In addition, we assume technological progress that implies a constant growth rate g of labour income. Hence, we can write

$$dz = gzdt - \Delta dq_w + \Delta dq_b,$$

where $\Delta \equiv w - b$. Job destruction takes place at an exogenous, state-dependent, arrival rate $s(z)$. The corresponding Poisson process counts how often our household moved from employment into unemployment which is q_w. Job creation takes place at an exogenous rate $\lambda(z)$ which is related to the matching function presented in (5.6.17). The Poisson process related to the matching process is denoted by q_b. It counts how often a household leaves its "b-status", i.e. how often a job is found. As an individual cannot lose his job when he does not have one and as finding a job makes (in this setup) no sense for someone who has a job, both arrival rates are state dependent. As an example, when an individual is employed, $\lambda(w) = 0$, when he is unemployed, $s(b) = 0$.

z	w	b
$\lambda(z)$	0	λ
$s(z)$	s	0

Table 11.2.1 *State dependent arrival rates*

Let the individual maximize expected utility $E_t \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau))\, d\tau$, where instantaneous utility is of the CES type, $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ with $\sigma > 0$.

11.2.2 The Bellman equation and value functions

The state space is described by a and z. The Bellman equation has the same structure as in (11.1.4). The adjustments that need to be made here follow from the fact that we have two state variables instead of one. Hence, the basic structure from (11.1.4) adopted to our problem reads $\rho V(a,z) = \max_c \left\{u(c) + \frac{1}{dt}E_t dV(a,z)\right\}$. The change of $V(a,z)$ is, given the evolution of a and z from above and the CVF from (10.2.11),

$$\begin{aligned} dV(a,z) = &\left\{V_a[ra + z - c] + V_z gz\right\}dt \\ &+ \left\{V(a, z-\Delta) - V(a,z)\right\}dq_w + \left\{V(a,z+\Delta) - V(a,z)\right\}dq_b. \end{aligned}$$

Forming expectations, remembering that $E_t dq_w = s(z)\,dt$ and $E_t dq_b = \lambda(z)\,dt$, and "dividing" by dt gives the Bellman equation

$$\rho V(a,z) = \max_c \left\{ \begin{array}{c} u(c) + V_a[ra + z - c] + V_z gz \\ +s(z)[V(a,z-\Delta) - V(a,z)] + \lambda(z)[V(a,z+\Delta) - V(a,z)] \end{array} \right\}. \tag{11.2.1}$$

The value functions in the matching literature are all special cases of this general Bellman equation.

Denote by $U \equiv V(a,b)$ the expected present value of (optimal behaviour of a worker) being unemployed (as in Pissarides, 2000, ch. 1.3) and by $W \equiv V(a,w)$, the expected present value of being employed. As the probability of losing a job for an unemployed worker is zero, $s(b) = 0$, and $\lambda(b) = \lambda$, the Bellman equation (11.2.1) reads

$$\rho U = \max_c \left\{u(c) + U_a[ra + b - c] + U_z gb + \lambda[W - U]\right\},$$

where we used that $W = V(a, b+\Delta)$. When we assume that agents behave optimally, i.e. we replace control variables by their optimal values, we obtain the maximized Bellman equation,

$$\rho U = u(c) + U_a[ra + b - c] + U_z gb + \lambda[W - U].$$

When we now assume that households can not save, i.e. $c = ra + b$, and that there is no technological progress, $g = 0$, we obtain

$$\rho U = u(ra + b) + \lambda[W - U].$$

Assuming further that households are risk-neutral, i.e. $u(c) = c$, and that they have no capital income, i.e. $a = 0$, consumption is identical to unemployment benefits $c = b$. If the interest rate equals the time preference rate, we obtain eq. (1.10) in Pissarides (2000),

$$rU = b + \lambda[W - U].$$

11.3 Intertemporal utility maximization under Brownian motion

11.3.1 The setup

Consider an individual whose budget constraint is given by

$$da = \{ra + w - pc\} dt + \beta a dz.$$

The notation is as always, uncertainty stems from Brownian motion z. The individual maximizes a utility function as given in (11.1.1), $U(t) = E_t \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau)) d\tau$. The value function is defined by $V(a) = \max_{\{c(\tau)\}} U(t)$ subject to the constraint. We again follow the three step scheme for dynamic programming.

11.3.2 Solving by dynamic programming

- DP1: Bellman equation and first-order conditions

The Bellman equation is given for Brownian motion by (11.1.4) as well. When a maximization problem other than one where (11.1.4) is suitable is to be formulated and solved, five adjustments are, in principle, possible for the Bellman equation. First, the discount factor ρ might be given by some other factor - for example the interest rate r when the present value of some firm is maximized. Second, the number of arguments of the value function needs to be adjusted to the number of state variables. Third, the number of control variables depends on the problem that is to be solved and, fourth, the instantaneous utility function is replaced by what is found in the objective function - which might be, for example, instantaneous profits. Finally and obviously, the differential $dV(.)$ needs to be computed according to the rules that are appropriate for the stochastic processes which drive the state variables.

As the differential of the value function, following Ito's Lemma in (10.2.3), is given by

$$dV(a) = \left\{ V'(a)[ra + w - pc] + \frac{1}{2} V''(a) \beta^2 a^2 \right\} dt + V'(a) \beta a dz,$$

forming expectations E_t and dividing by dt yields the Bellman equation for our specific problem

$$\rho V(a) = \max_{c(t)} \left\{ u(c(t)) + V'(a)[ra + w - pc] + \frac{1}{2} V''(a) \beta^2 a^2 \right\}$$

and the first-order condition is

$$u'(c(t)) = V'(a) p(t). \tag{11.3.1}$$

- DP2: Evolution of the costate variable

(i) The derivative of the maximized Bellman equation with respect to the state variable gives (using the envelope theorem) an equation describing the evolution of the costate variable,

$$\rho V' = V'' [ra + w - pc] + V'r + \frac{1}{2} V''' \beta^2 a^2 + V'' \beta^2 a \Leftrightarrow$$
$$(\rho - r) V' = V'' [ra + w - pc] + \frac{1}{2} V''' \beta^2 a^2 + V'' \beta^2 a. \qquad (11.3.2)$$

Not surprisingly, given that the Bellman equation already contains the second derivative of the value function, the derivative of the maximized Bellman equation contains its third derivative V'''.

(ii) Computing the differential of the shadow price of wealth $V'(a)$ gives, using Ito's Lemma (10.2.3),

$$dV' = V'' da + \frac{1}{2} V''' \beta^2 a^2 dt$$
$$= V'' [ra + w - pc] dt + \frac{1}{2} V''' \beta^2 a^2 dt + V'' \beta a dz,$$

and inserting into the partial derivative (11.3.2) of the maximized Bellman equation yields

$$dV' = (\rho - r) V' dt - V'' \beta^2 a dt + V'' \beta a dz$$
$$= \left\{ (\rho - r) V' - V'' \beta^2 a \right\} dt + V'' \beta a dz. \qquad (11.3.3)$$

As always at the end of DP2, we have a differential equation (or difference equation in discrete time) which determines the evolution of $V'(a)$, the shadow price of wealth.

- DP3: Inserting first-order conditions

Assuming that the evolution of aggregate prices is independent of the evolution of the marginal value of wealth, we can write the first-order condition for consumption in (11.3.1) as $du'(c) = pdV' + V'dp$. This follows, for example, from Ito's Lemma (10.2.6) with $\rho_{pV'} = 0$. Using (11.3.3) to replace dV', we obtain

$$du'(c) = p \left[\left\{ (\rho - r) V' - V'' \beta^2 a \right\} dt + V'' \beta a dz \right] + V' dp$$
$$= \left\{ (\rho - r) u'(c) - u''(c) c'(a) \beta^2 a \right\} dt + u''(c) c'(a) \beta a dz + u'(c) dp/p, \quad (11.3.4)$$

where the second equality uses the first-order condition $u'(c(t)) = V'p(t)$ to replace V' and the partial derivative of this first-order condition with respect to assets, $u''(c) c'(a) = V''p$, to replace V''.

When comparing this with the expression in (11.1.9) where uncertainty stems from a Poisson process, we see two common features: First, both Keynes-Ramsey rules have a stochastic term, the dz-term here and the dq-term in the Poisson case. Second, uncertainty affects the trend term for consumption in both terms. Here, this term contains the second derivative of the instantaneous utility function and $c'(a)$, in the Poisson case, we have the $\tilde{c}$ terms. The additional dp-term here stems from the assumption that prices are not constant. Such a term would also be visible in the Poisson case with flexible prices.

11.3.3 The Keynes-Ramsey rule

Just as in the Poisson case, we want a rule for the evolution of consumption here as well. We again define an inverse function and end up in the general case with

$$-\frac{u''}{u'}dc = \left\{r-\rho+\frac{u''(c)}{u'(c)}c_a\beta^2 a+\frac{1}{2}\frac{u'''(c)}{u'(c)}[c_a\beta a]^2\right\}dt \qquad (11.3.5)$$
$$-\frac{u''(c)}{u'(c)}c_a\beta a dz-\frac{dp}{p}.$$

With a CRRA utility function, we can replace the first, second and third derivative of $u(c)$ and find the corresponding rule

$$\sigma\frac{dc}{c} = \left\{r-\rho-\sigma\frac{c_a}{c}\beta^2 a+\frac{1}{2}\sigma[\sigma+1]\left[\frac{c_a}{c}\beta a\right]^2\right\}dt \qquad (11.3.6)$$
$$+\sigma\frac{c_a}{c}\beta a dz-\frac{dp}{p}.$$

11.4 Capital asset pricing

Let us again consider a typical CAP problem. This follows and extends Merton (1990, ch. 15; 1973). The presentation is in a simplified way.

11.4.1 The setup

The basic structure of the setup is identical as before. There is an objective function and a constraint. The objective function captures the preferences of our agent and is described by a utility function as in (11.1.1). The constraint is given by a budget constraint which will now be derived, following the principles of ch. 10.3.2.

Wealth a of households consist of a portfolio of assets i

$$a=\Sigma_{i=1}^{N}p_i n_i,$$

where the price of an asset is denoted by p_i and the number of shares held, by n_i. The total number of assets is given by N. Let us assume that the price of an asset follows geometric Brownian motion,

$$\frac{dp_i}{p_i}=\alpha_i dt+\sigma_i dz_i, \qquad (11.4.1)$$

where each price is driven by its own drift parameter α_i and its own variance parameter σ_i. Uncertainty results from Brownian motion z_i which is also specific for each asset. These parameters are exogenously given to the household but would in a general equilibrium setting be determined by properties of, for example, technologies, preferences and other parameters of the economy.

Households can buy or sell assets by using a share χ_i of their savings,

$$dn_i = \frac{1}{p_i}\chi_i \left\{\Sigma_{i=1}^N \pi_i n_i + w - c\right\} dt. \tag{11.4.2}$$

When savings are positive and a share χ_i is used for asset i, the number of stocks held in i increases. When savings are negative and χ_i is positive, the number of stocks i decreases.

The change in the households wealth is given by $da = \Sigma_{i=1}^N d(p_i n_i)$. The wealth held in one asset changes according to

$$d(p_i n_i) = p_i dn_i + n_i dp_i = \chi_i \left\{\Sigma_{i=1}^N \pi_i n_i + w - c\right\} dt + n_i dp_i.$$

The first equality uses Ito's Lemma from (10.2.6), taking into account that second derivatives of $F(.) = p_i n_i$ are zero and that dn_i in (11.4.2) is deterministic and therefore $dp_i dn_i = 0$. Using the pricing rule (11.4.1) and the fact that shares add to unity, $\Sigma_{i=1}^N \chi_i = 1$, the budget constraint of a household therefore reads

$$\begin{aligned} da &= \left\{\Sigma_{i=1}^N \pi_i n_i + w - c\right\} dt + \Sigma_{i=1}^N n_i p_i \left[\alpha_i dt + \sigma_i dz_i\right] \\ &= \left\{\Sigma_{i=1}^N \frac{\pi_i}{p_i} n_i p_i + n_i p_i \alpha_i + w - c\right\} dt + \Sigma_{i=1}^N n_i p_i \sigma_i dz_i \\ &= \left\{\Sigma_{i=1}^N a_i \left[\frac{\pi_i}{p_i} + \alpha_i\right] + w - c\right\} dt + \Sigma_{i=1}^N a_i \sigma_i dz_i. \end{aligned}$$

Now define θ_i as always as the share of wealth held in asset i, $\theta_i \equiv a_i/a$. Then, by definition, $a = \Sigma_{i=1}^N \theta_i a$ and shares add up to unity, $\Sigma_{i=1}^N \theta_i = 1$. We rewrite this for later purposes as

$$\theta_N = 1 - \Sigma_{i=1}^{N-1} \theta_i \tag{11.4.3}$$

Define further the interest rate for asset i and the interest rate of the market portfolio by

$$r_i \equiv \frac{\pi_i}{p_i} + \alpha_i, \qquad r \equiv \Sigma_{i=1}^N \theta_i r_i. \tag{11.4.4}$$

This gives us the budget constraint,

$$\begin{aligned} da &= \left\{a\Sigma_{i=1}^N \theta_i r_i + w - c\right\} dt + a\Sigma_{i=1}^N \theta_i \sigma_i dz_i \\ &= \left\{ra + w - c\right\} dt + a\Sigma_{i=1}^N \theta_i \sigma_i dz_i. \end{aligned} \tag{11.4.5}$$

11.4.2 Optimal behaviour

Let us now consider an agent who behaves optimally when choosing her portfolio and in making her consumption-saving decision. We will not go through all the steps to derive a Keynes-Ramsey rule as asset pricing requires only the Bellman equation and first-order conditions.

- The Bellman equation

The Bellman equation is given by (11.1.4), i.e. $\rho V(a) = \max_{c(t),\theta_i(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a) \right\}$. Hence, we need again the expected change of the value of one unit of wealth. With one state variable, we simply apply Ito's Lemma from (10.2.1) and find

$$\frac{1}{dt} E_t dV(a) = \frac{1}{dt} E_t \left(V'(a)\, da + \frac{1}{2} V''(a) [da]^2 \right). \tag{11.4.6}$$

In a first step required to obtain the explicit version of the Bellman equation, we compute the square of da. It is given, taking (10.2.2) into account, by $[da]^2 = a^2 \left[\Sigma_{i=1}^N \theta_i \sigma_i dz_i \right]^2$. When we compute the square of the sum, the expression for the product of Brownian motions in (10.2.5) becomes important as correlation coefficients need to be taken into consideration. Denoting the covariances by $\sigma_{ij} \equiv \sigma_i \sigma_j \rho_{ij}$, we get

$$\begin{aligned} [da]^2 &= a^2 \left[\theta_1 \sigma_1 dz_1 + \theta_2 \sigma_2 dz_2 + ... + \theta_n \sigma_n dz_n \right]^2 \\ &= a^2 \left[\theta_1^2 \sigma_1^2 dt + \theta_1 \sigma_1 \theta_2 \sigma_2 \rho_{12} dt + ... + \theta_2 \sigma_2 \theta_1 \sigma_1 \rho_{12} dt + \theta_2^2 \sigma_2^2 dt + ... + ... \right] \\ &= a^2 \Sigma_{j=1}^N \Sigma_{i=1}^N \theta_i \theta_j \sigma_{ij} dt. \end{aligned} \tag{11.4.7}$$

Now rewrite the sum in (11.4.7) as follows

$$\begin{aligned} \Sigma_{j=1}^N \Sigma_{i=1}^N \theta_j \theta_i \sigma_{ij} &= \Sigma_{j=1}^{N-1} \Sigma_{i=1}^N \theta_j \theta_i \sigma_{ij} + \Sigma_{i=1}^N \theta_i \theta_N \sigma_{iN} \\ &= \Sigma_{j=1}^{N-1} \Sigma_{i=1}^{N-1} \theta_j \theta_i \sigma_{ij} + \Sigma_{i=1}^N \theta_i \theta_N \sigma_{iN} + \Sigma_{j=1}^{N-1} \theta_N \theta_j \sigma_{Nj} \\ &= \Sigma_{j=1}^{N-1} \Sigma_{i=1}^{N-1} \theta_j \theta_i \sigma_{ij} + \Sigma_{i=1}^{N-1} \theta_i \theta_N \sigma_{iN} + \Sigma_{j=1}^{N-1} \theta_N \theta_j \sigma_{Nj} + \theta_N^2 \sigma_N^2 \\ &= \Sigma_{j=1}^{N-1} \Sigma_{i=1}^{N-1} \theta_j \theta_i \sigma_{ij} + 2 \Sigma_{i=1}^{N-1} \theta_i \theta_N \sigma_{iN} + \theta_N^2 \sigma_N^2 \end{aligned}$$

As the second term, using (11.4.3), can be written as $\Sigma_{i=1}^{N-1} \theta_i \theta_N \sigma_{iN} = \left[1 - \Sigma_{j=1}^{N-1} \theta_i \right] \Sigma_{i=1}^{N-1} \theta_i \sigma_{iN}$, our $(da)^2$ reads

$$\begin{aligned} (da)^2 &= a^2 \left\{ \Sigma_{j=1}^{N-1} \Sigma_{i=1}^{N-1} \theta_j \theta_i \sigma_{ij} + 2 \left[1 - \Sigma_{j=1}^{N-1} \theta_i \right] \Sigma_{i=1}^{N-1} \theta_i \sigma_{iN} + \left[1 - \Sigma_{j=1}^{N-1} \theta_i \right]^2 \sigma_N^2 \right\} dt \\ &= a^2 \left\{ \Sigma_{j=1}^{N-1} \Sigma_{i=1}^{N-1} \theta_j \theta_i (\sigma_{ij} - 2\sigma_{iN}) + 2 \Sigma_{i=1}^{N-1} \theta_i \sigma_{iN} + \left[1 - \Sigma_{i=1}^{N-1} \theta_i \right]^2 \sigma_N^2 \right\} dt. \end{aligned} \tag{11.4.8}$$

The second preliminary step for obtaining the Bellman equation uses (11.4.3) again and expresses the interest rate from (11.4.4) as a sum of the interest rate of asset N (which could but does not need to be a riskless asset) and weighted excess returns $r_i - r_N$,

$$r = \Sigma_{i=1}^{N-1} \theta_i r_i + \left(1 - \Sigma_{i=1}^{N-1} \theta_i \right) r_N = r_N + \Sigma_{i=1}^{N-1} \theta_i \left[r_i - r_N \right]. \tag{11.4.9}$$

The Bellman equation with (11.4.8) and (11.4.9) now finally reads

$$\rho V(a) = \max_{c(t),\theta_i(t)} \left\{ u(c) + V'(a) \left[(r_N + \Sigma_{i=1}^{N-1} \theta_i [r_i - r_N]) a + w - c \right] + \frac{1}{2} V''(a) [da]^2 \right\},$$

where $(da)^2$ should be thought of representing (11.4.8).

- First-order conditions

The first-order conditions are the first-order condition for consumption,

$$u'(c) = V'(a),$$

and the first-order condition for assets. The first-order condition for consumption has the well-known form.

To compute first-order conditions for shares θ_i, we compute $d\{.\}/d\theta_i$ for (11.4.8),

$$\begin{aligned}\frac{d\{.\}}{d\theta_i} &= \Sigma_{j=1}^{N-1}\theta_j(\sigma_{ij} - 2\sigma_{iN}) + \Sigma_{i=1}^{N-1}\theta_i(\sigma_{ij} - 2\sigma_{iN}) + 2\sigma_{iN} - 2\left[1 - \Sigma_{j=1}^{N-1}\theta_i\right]\sigma_N^2 \\ &= 2\left\{\Sigma_{j=1}^{N-1}\theta_j(\sigma_{ij} - 2\sigma_{iN}) + \sigma_{iN} - \left[1 - \Sigma_{j=1}^{N-1}\theta_i\right]\sigma_N^2\right\} \\ &= 2\left\{\Sigma_{j=1}^{N-1}\theta_j(\sigma_{ij} - \sigma_{iN}) + \left[1 - \Sigma_{j=1}^{N-1}\theta_i\right]\left(\sigma_{iN} - \sigma_N^2\right)\right\} \\ &= 2\Sigma_{j=1}^{N}\theta_i\left[\sigma_{ij} - \sigma_{iN}\right]. \end{aligned} \tag{11.4.10}$$

Hence, the derivative of the Bellman equation with respect to θ_i is with (11.4.9) and (11.4.10)

$$V'a[r_i - r_N] + \frac{1}{2}V''2a^2\Sigma_{j=1}^{N}\theta_i[\sigma_{ij} - \sigma_{iN}] = 0 \Leftrightarrow$$

$$r_i - r_N = -\frac{V''}{V'}a\Sigma_{j=1}^{N}\theta_i\left[\sigma_{ij} - \sigma_{iN}\right]. \tag{11.4.11}$$

The interpretation of this optimality rule should take into account that we assumed that an interior solution exists. This condition, therefore, says that agents are indifferent between the current portfolio and a marginal increase in a share θ_i if the difference in instantaneous returns, $r_i - r_N$, is compensated by the covariances of of assets i and N. Remember that from (11.4.4), instantaneous returns are certain at each instant.

11.4.3 Capital asset pricing

Given optimal behaviour of agents, we now derive the well-known capital asset pricing equation. Start by assuming that asset N is riskless, i.e. $\sigma_N = 0$ in (11.4.1). This implies that it has a variance of zero and therefore a covariance σ_{Nj} with any other asset of zero as well, $\sigma_{Nj} = 0\ \forall j$. Define further $\gamma \equiv -a\frac{V''}{V'}$, the covariance of asset i with the market portfolio as $\sigma_{iM} \equiv \Sigma_{j=1}^{N}\theta_j\sigma_{ij}$, the variance of the market portfolio as $\sigma_M^2 \equiv \Sigma_{j=1}^{N}\theta_i\sigma_{iM}$ and the return of the market portfolio as $r \equiv \Sigma_{i=1}^{N}\theta_i r_i$ as in (11.4.4).

We are now only few steps away from the CAP equation. Using the definition of γ and σ_{iM} allows to rewrite the first-order condition for shares (11.4.11) as

$$r_i - r_N = \gamma\sigma_{iM}. \tag{11.4.12}$$

Multiplying this first-order condition by the share θ_i gives $\theta_i[r_i - r_N] = \theta_i\gamma\sigma_{iM}$. Summing up all assets, i.e. applying $\Sigma_{j=1}^{N}$ to both sides, and using the above definitions yields

$$r - r_N = \gamma\sigma_M^2.$$

Dividing this expression by version (11.4.12) of the first-order condition yields the capital asset pricing equation,

$$r_i - r_N = \frac{\sigma_{iM}}{\sigma_M^2} (r - r_N) .$$

The ratio σ_{iM}/σ_M^2 is what is usually called the β-factor.

11.5 Natural volatility II

Before this book comes to an end, the discussion of natural volatility models in ch. 8.4 is completed in this section. We will present a simplified version of those models that appear in the literature which are presented in stochastic continuous time setups. The usefulness of Poisson processes will become clear here. Again, more background is available on http://www.waelde.com/nv.html.

11.5.1 An real business cycle model

This section presents the simplest general equilibrium setup that allows to study fluctuations stemming from occasional jumps in technology. The basic belief is that economically relevant changes in technologies are rare and occur every 5-8 years. Jumps in technology means that the technological level, as captured by the TFP level, increases. Growth cycles therefore result without any negative TFP shocks.

- Technologies

The economy produces a final good by using a Cobb-Douglas technology

$$Y = K^{\alpha} (AL)^{1-\alpha} . \tag{11.5.1}$$

Total factor productivity is modelled as labour augmenting labour productivity. While this is of no major economic importance given the Cobb-Douglas structure, it simplifies the notation below. Labour productivity follows a geometric Poisson process with drift

$$dA/A = gdt + \gamma dq, \tag{11.5.2}$$

where g and γ are positive constants and λ is the exogenous arrival rate of the Poisson process q. We know from (10.5.17) in ch. 10.5.4 that the growth rate of the expected value of A is given by $g + \lambda\gamma$.

The final good can be used for consumption and investment, $Y = C + I$, which implies that the prices of all these goods are identical. Choosing Y as the numéraire good, the price is one for all these goods. Investment increases the stock of production units K if investment is larger than depreciation, captured by a constant depreciation rate δ,

$$dK = (Y - C - \delta K) \, dt. \tag{11.5.3}$$

There are firms who maximize instantaneous profits. They do not bear any risk and pay factors r and w, marginal productivities of capital and labour.

- Households

Households maximize their objective function

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\, d\tau$$

by choosing the consumption path $\{c(\tau)\}$. Instantaneous utility can be specified by

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}. \tag{11.5.4}$$

Wealth of households consists of shares in firms which are denoted by k. This wealth changes in a deterministic way (we do not derive it here but it could be done following the steps in ch. 10.3.2), despite the presence of TFP uncertainty. This is due to two facts: First, wealth is measured in units of physical capital, i.e. summing k over all households gives K. As the price of one unit of K equals the price of one unit of the output good and the latter was chosen as numéraire, the price of one unit of wealth is non-stochastic. This differs from (10.3.8) where the price jumps when q jumps. Second, a jump in q does not affect k directly. This could be the case when new technologies make part of the old capital stock obsolete. Hence, the constraint of households is a budget constraint which reads

$$dk = (rk + w - c)\, dt. \tag{11.5.5}$$

The interest rate is given by the difference between the marginal product of capital and the depreciation rate, $r = \partial Y / \partial K - \delta$.

- Optimal behaviour

When computing optimal consumption levels, households take the capital stock k and the TFP level A as their state variables into account. This setup is therefore similar to the deterministic two-state maximization problem in ch. 6.3. Going through similar steps (concerning, for example, the substituting of cross derivatives in step DP2) and taking the specific aspects of this stochastic framework into account, yields following optimal consumption (see exercise 8)

$$\sigma \frac{dc}{c} = \left\{ r - \rho + \lambda \left[\left(\frac{c}{\tilde{c}} \right)^\sigma - 1 \right] \right\} dt + \left\{ \frac{\tilde{c}}{c} - 1 \right\} dq. \tag{11.5.6}$$

Despite the deterministic constraint (11.5.5) and due to TFP jumps in (11.5.2), consumption jumps as well: a $dq-$term shows up in this expression and marginal utility levels before $(c^{-\sigma})$ and after $(\tilde{c}^{-\sigma})$ the jump, using the notation from (10.1.7) appear as well. Marginal utilities appear in the deterministic part of this rule due to precautionary saving considerations. The reason for the jump is straightforward: whenever there is a discrete increase in the TFP level, the interest rate and wages jump. Hence, returns for savings or households change and the household adjusts its consumption level. This is in principle identical to the behaviour in the deterministic case as illustrated in fig. 5.6.1 in ch. 5.6.1. (Undertaking this here for this stochastic case would be very useful.)

- General equilibrium

We are now in a position to start thinking about the evolution of the economy as a whole. It is described by a system in three equations. Labour productivity follows (11.5.2). The capital stock follows

$$dK = \left(K^{\alpha}(AL)^{1-\alpha} - C - \delta K\right) dt$$

from (11.5.1) and (11.5.3). Aggregate consumption follows

$$\sigma \frac{dC}{C} = \left\{ \alpha \left[\frac{AL}{K}\right]^{1-\alpha} - \delta - \rho + \lambda \left[\left(\frac{C}{\tilde{C}}\right)^{\sigma} - 1\right] \right\} dt + \left\{ \frac{\tilde{C}}{C} - 1 \right\} dq$$

from aggregating over households using (11.5.6). Individual consumption c is replaced by aggregate consumption C and the interest rate is expressed by marginal productivity of capital minus depreciation rate. This system looks fairly similar to deterministic models, the only substantial difference lies in the dq term and the post-jump consumption levels $\tilde{C}$.

- Equilibrium properties for a certain parameter set

The simplest way to get an intuition about how the economy evolves consists in looking at an example, i.e. by looking at a solution of the above system that holds for a certain parameter set. We choose as example the parameter set for which the saving rate is constant and given by $s = 1 - \sigma^{-1}$. The parameter set for which $C = (1-s)Y$ is optimal, is given by $\sigma = \left(\rho + \delta - \lambda\left[(1+\gamma)^{1-\alpha} - 1\right]\right) / \left(\alpha\delta - (1-\alpha)g\right)$. As we need $\sigma > 1$ for a meaningful saving rate, the intertemporal elasticity of substitution σ^{-1} is smaller than one. For a derivation of this result, see the section on "Closed-form solutions with parameter restrictions" in "further reading".

The dynamics of capital and consumption can then be best analyzed by looking at auxiliary variables, in this case capital per effective worker, $\hat{K} = K/A$. This auxiliary variable is needed to remove the trend out of capital K. The original capital stock grows without bound, the auxiliary variable has a finite range. Using auxiliary variables of this type has a long tradition in growth models as detrending is a common requirement for an informative analysis. The evolution of this auxiliary variable is given by (applying the appropriate CVF)

$$d\hat{K} = \left\{\beta_1 \hat{K}^{\alpha} L^{1-\alpha} - \beta_2 \hat{K}\right\} dt - \beta_3 \hat{K} dq, \tag{11.5.7}$$

where β_i are functions of preference and technology parameters of the model. One can show that $0 < \beta_3 < 1$. The evolution of this capital stock per effective worker can be illustrated by using a figure which is similar to those used to explain the Solow growth model. The following figure shows the evolution of the capital stock per worker on the left and the evolution of GDP on the right.

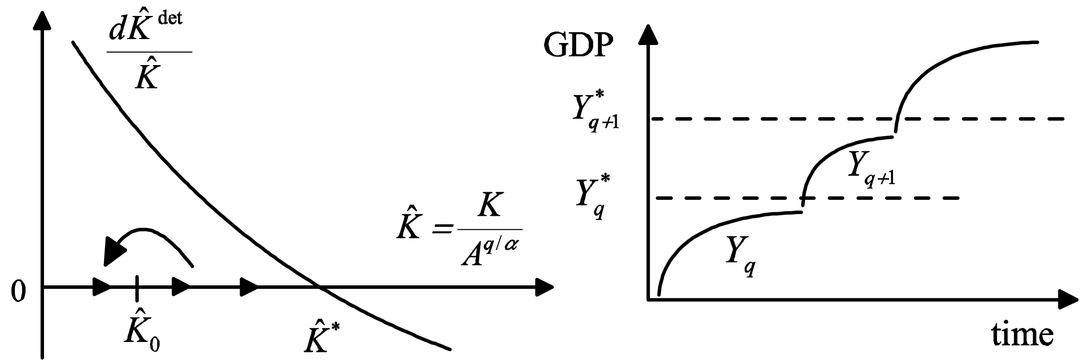


Figure 11.5.1 *Cyclical growth*

Assume the initial capital stock $\hat{K}$ is given by $\hat{K}_0$. Assume also, for the time being, that there is no technology jump, i.e. $dq = 0$. The capital stock $\hat{K}$ then increases smoothly over time and approaches a temporary steady state $\hat{K}^*$ which follows from (11.5.7) with $dq = 0$ and $d\hat{K} = 0$. This temporary steady state is given by $\hat{K}^* = \left(\beta_1 L^{1-\alpha}/\beta_2\right)^{1/(1-\alpha)}$ and has properties in perfect analogy to the steady state in the Solow model. When a technology jump occurs, the capital stock per effective worker diminishes as the denominator in $\hat{K} = K/A$ increases but the capital stock in the numerator in the instant of the jump does not change. The capital stock per effective worker is "thrown back", as indicated by the arrow in the left panel above. After that, the capital stock $\hat{K}$ again approaches the temporary steady state $\hat{K}^*$.

The right panel of the above figure shows what these technology jumps do to GDP. As long as there is no technology jump, GDP approaches an upper limit Y_q^* which is specific to the technology level q. A technology jump increases GDP Y_q instantaneously as TFP goes up. (This increase could be very small, depending on what share of production units enjoy an increase in TFP.) The more important increase, however, results from the the shift in the upper limit from Y_q^* to Y_{q+1}^*. Capital accumulation following the technology jump increases TFP which now approaches the new upper limit. This process of endogenous growth cycles continues ad infinitum.

11.5.2 A natural volatility model

The above analysis can be extended to allow for endogenous technology jumps. As in the discrete time version of this setup, the probability that a technology jump occurs is made a function of resources R invested into R&D. In contrast to (8.4.3), however, it is the arrival rate and not the probability itself which is a function of R,

$$\lambda = \lambda(R).$$

This is a requirement of continuous time setups and is builds on a long tradition in continuous time models with Poisson processes. The resource constraint of the economy is then extended accordingly to

$$dK = (Y - C - R - \delta K)\, dt$$

by including the resources R.

A model of this type can then be solved as before. Either one considers special parameter values and obtains closed-form solutions or one performs a numerical analysis (see below). The qualitative properties of cyclical growth are identical to the ones presented before in fig. 11.5.1. The crucial economic difference consists in the fact that the frequency of technology jumps, i.e. the average length of a business cycle, depend on decisions of households. If households find it profitable to shift a larger amount of resources into R&D, business cycles will be shorter. Not only the long-run growth rate, but also short-run fluctuations are influenced by fundamental parameters of the model as also by government policy. All these questions are analyzed in detail in the "natural volatility" literature.

11.5.3 A numerical approach

It is helpful for a numerical solution to have a model description in stationary variables. To this end, define auxiliary variables $\hat{K} = K/A$ and $\hat{C} = C/A$. The one for capital is the same as was used in (11.5.7), the one for consumption is new as we will now not work with a closed-form solution. Let us look at a situation with exogenous arrival rates, i.e. at the RBC model of above, to illustrate the basic approach for a numerical solution. When computing the dynamics of these variables (see "further reading" for references), we find

$$\sigma \frac{d\hat{C}}{\hat{C}} = \left\{ \alpha \left(\frac{L}{\hat{K}} \right)^{1-\alpha} - \delta - \rho - \sigma g + \lambda \left[\left(\frac{\hat{C}}{(1+\gamma)\tilde{\hat{C}}} \right)^{\sigma} - 1 \right] \right\} dt + \sigma \left\{ \frac{\tilde{\hat{C}}}{\hat{C}} - 1 \right\} dq, \tag{11.5.8}$$

$$d\hat{K} = \left\{ \hat{Y} - (\delta + g)\hat{K} - \hat{C} \right\} dt - \frac{\gamma}{1+\gamma} \hat{K} dq. \tag{11.5.9}$$

When we look at these equations, they "almost" look like ordinary differential equations. The only difficulty is contained in the term $\tilde{\hat{C}}$. To understand the solution procedure, think of the saddle-path trajectory in the optimal growth model - see fig. 5.6.2 in ch. 5.6.3. Given the transformation undertaken here, the solution of this system is given by a policy function $\hat{C}\left(\hat{K}\right)$ which is *not* a function of the technological level A. The objective of the solution therefore consists of finding this $\hat{C}\left(\hat{K}\right)$, a saddle path in analogy to the one in fig. 5.6.2. The term $\tilde{\hat{C}}$ stands for the consumption level after the jump. As the functional relationship is independent of A and thereby the number of jumps, we can write $\tilde{\hat{C}} = \hat{C}\left(\frac{1}{1+\gamma}\hat{K}\right)$, i.e. it is the consumption level at the capital stock $\frac{1}{1+\gamma}\hat{K}$ after a jump. As $\gamma > 0$, a jump implies a reduction in the auxiliary, i.e. technology-transformed, capital stock $\hat{K}$.

The "trick" in solving this system now consists of providing not only two initial conditions but one initial condition for capital an initial *path* for consumption $\hat{C}\left(\hat{K}\right)$. This initial path is then treated as an exogenous variable in the denominator of (11.5.8) and the above differential equations have been transformed into a system of ordinary differential equations! Letting the initial paths start at the origin and making them linear, the question then simply consists of finding the right slope such that the solution of the above system identifies a saddle path and steady state. For more details and an implementation, see "Numerical solution" in "further reading".

11.6 Further reading and exercises

- Mathematical background on dynamic programming

There are many, and in most cases much more technical, presentations of dynamic programming in continuous time under uncertainty. A classic mathematical reference is Gihman and Skorohod (1972) and a widely-used mathematical textbook is Øksendal (1998); see also Protter (1995). These books are probably useful only for those wishing to work on the theory of optimization and not on applications of optimization methods. Kushner (1967) and Dempster (1991) have a special focus on Poisson processes. Optimization with unbounded utility functions by dynamic programming was studied by Sennewald (2007).

With SDEs we need boundary conditions as well. In the infinite horizon case, we would need a transversality condition (TVC). See Smith (1996) for a discussion of a TVC in a setup with Epstein-Zin preferences. Sennewald (2007) has TVCs for Poisson uncertainty.

- Applications

Books in finance that use dynamic programming methods include Duffie (1988, 2001) and Björk (2004). Stochastic optimization for Brownian motion is also covered nicely in Chang (2004).

A maximization problem of the type presented in ch. 11.1 was first analyzed in Wälde (1999, 2008). This chapter combines these two papers. These two papers were also jointly used in the Keynes-Ramsey rule appendix to Wälde (2005). It is also used in Posch and Wälde (2006), Sennewald and Wälde (2006) and elsewhere.

Optimal control in stochastic continuous time setups is used in many applications. Examples include issues in international macro (Obstfeld, 1994, Turnovsky, 1997, 2000), international finance and debt crises (Stein, 2006) and also the analysis of the permanent-income hypothesis (Wang, 2006) or of the wealth distribution hypothesis (Wang, 2007), and many others. A firm maximization problem with risk-neutrality where R&D increases quality of goods, modelled by a stochastic differential equation with Poisson uncertainty, is presented and solved by Dinopoulos and Thompson (1998, sect. 2.3).

The Keynes-Ramsey rule in (11.3.4) was derived in a more or less complex framework by Breeden (1986) in a synthesis of his consumption based capital asset pricing model, Cox, Ingersoll and Ross (1985) in their continuous time capital asset pricing model, or Turnovsky (2000) in his textbook. Cf. also Obstfeld (1994).

There are various recent papers which use continuous time methods under uncertainty. For examples from finance and monetary economics, see DeMarzo and Urošević (2006), Gabaix et al. (2006), Maenhout (2006), Piazzesi (2005), examples from risk theory and learning include Kyle et al. (2006) and Keller and Rady (1999), for industrial organization see, for example, Murto (2004). In spatial economics there is, for example, Gabaix (1999), the behaviour of households in the presence of durable consumption goods is analyzed by Bertola et al. (2005), R&D dynamics are investigated by Bloom (2007) and mismatch and exit rates in labour economics are analyzed by Shimer (2007,2008). The effect of technological progress on unemployment is analyzed by Prat (2007). The real options approach to investment or hiring under uncertainty is another larger area. See, for example, Bentolila and Bertola (1990) or Guo et al. (2005). Further references to papers that use Poisson processes can be found in ch. 10.6.

- Closed form solutions

Closed-form solutions and analytical expressions for value functions have been derived by many authors. This approach was pioneered by Merton (1969, 1971) for Brownian motion. Chang (2004) devotes an entire chapter (ch. 5) to closed-form solutions for Brownian motion. For setups with Poisson-uncertainty, Dinopoulos and Thompson (1998, sect. 2.3), Wälde (1999b) or Sennewald and Wälde (2006, ch. 3.4) derive closed-form solutions. An overview is provided by Wälde (2011).

Closed-form solutions for Levy processes are available, for example, from Aase (1984), Framstad, Øksendal and Sulem (2001) and in the textbook by Øksendal and Sulem (2005).

- Closed-form solutions with parameter restrictions

Sometimes, restricting the parameter set of the economy in some intelligent way allows to provide closed-form solutions for very general models. These solutions provide insights which cannot be obtained that easily by numerical analysis. Early examples are Long and Plosser (1983) and Benhabib and Rustichini (1994) who obtain closed-form solutions for a discrete-time stochastic setup. In deterministic, continuous time, Xie (1991, 1994) and Barro, Mankiw and Sala-i-Martin (1995) use this approach as well. Wälde (2005) and Wälde and Posch (2006) derive closed-form solutions for business cycle models with Poisson uncertainty. Sennewald and Wälde (2006) study an investment and finance problem. The example in section 11.5.1 is taken from Schlegel (2004). The most detailed step-by-step presentation of the solution technique is in the Referees' appendix to Wälde (2005).

- Natural volatility

The expression "natural volatility" represents a certain view about why almost any economic time series exhibits fluctuations. Natural volatility says that fluctuations are natural, they are intrinsic to any growing economy. An economy that grows is also an economy that fluctuates. Growth and fluctuations are "two sides of the same coin", they have the same origin: new technologies.

Published papers in this literature are (in sequential and alphabetical order) Fan (1995), Bental and Peled (1996), Freeman, Hong and Peled (1999), Matsuyama (1999, 2001), Wälde (1999, 2002, 2005), Li (2001) Francois and Lloyd-Ellis (2003,2008), Maliar and Maliar (2004) and Phillips and Wrase (2006).

- Numerical solution

The numerical solution was analyzed and implemented by Schlegel (2004).

- Matching and saving

Ch. 11.2 shows where value functions in the matching literature come from. This chapter uses a setup where uncertainty in labour income is combined with saving. It thereby presents the typical setup of the saving and matching literature. The setup used here was explored in more detail by Bayer and Wälde (2010a, b) and Bayer et al. (2010,).

The matching and saving literature in general builds on incomplete market models where households can insure against income risk by saving (Huggett, 1993; Aiyagari, 1994; Huggett and Ospina, 2001; Marcet et al., 2007). First analyses of matching and saving include Andolfatto (1996) and Merz (1995) where individuals are fully insured against labour income risk as labour income is pooled in large families. Papers which exploit the advantage of CARA utility functions include Acemoglu and Shimer (1999), Hassler et al. (2005), Shimer and Werning, (2007, 2008) and Hassler and Rodriguez Mora, 1999, 2008). Their closed-form solutions for the consumption-saving decision cannot always rule out negative consumption levels for poor households. Bils et al. (2007, 2009), Nakajima (2008) and Krusell et al. (2010) work with a CRRA utility function in discrete time.

Exercises Chapter 11
Applied Intertemporal Optimization
Dynamic Programming in continuous time under uncertainty

1. Optimal saving under Poisson uncertainty with two state variables
   Consider the objective function

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau)) \, d\tau$$

   and the budget constraint

$$da(t) = \{ra(t) + w - p(t) c(t)\} dt + \beta a(t) dq(t),$$

   where r and w are constant interest and wage rates, $q(t)$ is a Poisson process with an exogenous arrival rate λ and β is a constant as well. Letting g and σ denote constants, assume that the price $p(t)$ of the consumption good follows

$$dp(t) = p(t) [gdt + \sigma dq(t)].$$

   (a) Derive a rule which optimally describes the evolution of consumption. Deriving this rule in the form of marginal utility, i.e. $du'(c(t))$ is sufficient.

   (b) Derive a rule for consumption, i.e. $dc(t) = ...$

   (c) Derive a rule for optimal consumption for $\beta = 0$ or $\sigma = 0$.

2. Optimal saving under Brownian motion
   Derive the Keynes-Ramsey rules in (11.3.5) and (11.3.6), starting from the rule for marginal utility in (11.3.4).

3. Adjustment cost
   Consider a firm that maximizes its present value defined by $\Pi = E_t \int_t^\infty e^{-r[\tau - t]} \pi(\tau) \, d\tau$. The firm's profit π is given by the difference between revenues and costs, $\pi = px - ci^2$, where output is assumed to be a function of the current capital stock, $x = k^\alpha$. The firm's control variable is investment i that determines its capital stock,

$$dk = (i - \delta k) \, dt.$$

The firm operates in an uncertain environment. Output prices and costs for investment evolve according to

$$dp/p = \alpha_p dt + \sigma_p dz_p, \quad dc/c = \alpha_c dt + \sigma_c dz_c,$$

where z_p and z_c are two independent stochastic processes.

(a) Assume z_p and z_c are two independent Brownian motions. Set $\alpha_p = \sigma_p = 0$, such that the price p is constant. What is the optimal investment behaviour of the firm?

(b) Consider the alternative case where costs c are constant but prices p follow the above SDE. How much would the firm now invest?

(c) Provide an answer to the question in a) when z_c is a Poisson process.

(d) Provide an answer to the question in b) when z_p is a Poisson process.

4. Firm specific technological progress
Consider a firm facing a demand function with price elasticity ε, $x = \Phi p^{-\varepsilon}$, where p is the price and Φ is a constant. Let the firm's technology be given by $x = a^q l$ where $a > 1$. The firm can improve its technology by investing in R&D. R&D is modelled by the Poisson process q which jumps with arrival rate $\lambda(l_q)$ where l_q is employment in the research department of the firm. The exogenous wage rate the firm faces amounts to w.

(a) What is the optimal static employment level l of this firm for a given technological level q?

(b) Formulate an intertemporal objective function given by the present value of the firms profit flows over an infinite time horizon. Continue to assume that q is constant and let the firm choose l optimally from this intertemporal perspective. Does the result change with respect to (a)?

(c) Using the same objective function as in (b), let the firm now determine both l and l_q optimally. What are the first-order conditions? Give an interpretation in words.

(d) Compute the expected output level for $\tau > t$, given the optimal employment levels l^* and l_q^*. In other words, compute $E_t x(\tau)$. Hint: Derive first a stochastic differential equation for output $x(t)$.

5. Budget constraints and optimal saving and finance decisions
Imagine an economy with two assets, physical capital $K(t)$ and government bonds $B(t)$. Let wealth $a(t)$ of households be given by $a(t) = v(t)k(t) + b(t)$ where $v(t)$ is the price of one unit of capital, $k(t)$ is the number of stocks and $b(t)$ is the

nominal value of government bonds held by the household. Assume the price of stocks follows

$$dv(t) = \alpha v(t)\,dt + \beta v(t)\,dq(t),$$

where α and β are constants and $q(t)$ is a Poisson process with arrival rate λ.

(a) Derive the budget constraint of the household. Use $\theta(t) \equiv v(t)\,k(t)/a(t)$ as the share of wealth held in stocks.

(b) Derive the budget constraint of the household by assuming that $q(t)$ is Brownian motion.

(c) Now let the household live in a world with three assets (in addition to the two above, there are assets available on $f$oreign markets). Assume that the budget constraint of the household is given by

$$da(t) = \{r(t)\,a(t) + w(t) - p(t)\,c(t)\}\,dt + \beta_k \theta_k(t)\,a(t)\,dq(t) + \beta_f \theta_f(t)\,a(t)\,dq_f(t),$$

where

$$r(t) = \theta_k(t)\,r_k + \theta_f(t)\,r_f + (1 - \theta_k(t) - \theta_f(t))\,r_b$$

is the interest rate depending on weights $\theta_i(t)$ and constant instantaneous interest rates r_k, r_f and r_b. Let $q(t)$ and $q_f(t)$ be two Poisson processes. Given the usual objective function

$$U(t) = E_t \int_t^{\infty} e^{-\rho[\tau - t]} u(c(\tau))\,d\tau,$$

what is the optimal consumption rule? What can be said about optimal shares θ_k^* and θ_f^*?

6. Capital asset pricing in ch. 11.4.3
The covariance of asset i with the market portfolio is denoted by σ_{iM}, the variance of the market portfolio is σ_M^2 and the return of the market portfolio is $r_M \equiv \Sigma_i \theta_i r_i$.

(a) Show that the covariance of asset i with the market portfolio σ_{iM} is given by $\Sigma_{j=1}^N \theta_j \sigma_{ij}$.

(b) Show that $\Sigma_{j=1}^N \theta_i \sigma_{iM}$ is the variance of the market portfolio σ_M^2.

7. Standard and non-standard technologies
Let the social welfare function of a central planner be given by

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau-t]} u(C(\tau)) \, d\tau.$$

   (a) Consider an economy where the capital stock follows $dK = AK^\alpha L^{1-\alpha} [\mu dt + \sigma dz] - (\delta K + C) dt$ where dz is the increment of Brownian motion and μ and σ are constants. Derive the Keynes-Ramsey rule for this economy.
   (b) Assume that $dY = \Theta [\mu dt + \sigma dz]$ and that Θ is constant. What is the expected level of $Y(\tau)$ for $\tau > t$, i.e. $E_t Y(\tau)$?
   (c) Consider an economy where the technology is given by $Y = AK^\alpha L^{1-\alpha}$ with $dA = \gamma A dt + \beta A dz$, where z is Brownian motion. Let the capital stock follow $dK = (Y - \delta K - C) dt$. Derive the Keynes-Ramsey rule for this economy as well.
   (d) Is there a parameter constellation under which the Keynes-Ramsey rules are identical?

8. Standard and non-standard technologies II
Provide answers to the same questions as in "Standard and non-standard technologies" but assume that z is a Poisson process with arrival rate λ. Compare your result to (11.5.6).

Chapter 12

Miscellanea, references and index

The concept of time

What is time? Without getting into philosophical details, it is useful to be precise here about how time is denoted. Time is always denoted by τ. Time can take different values. The most important (frequently encountered) one is the value t where t stands for $t$oday. This is true both for the discrete and for the continuous time context. In a discrete time context, $t+1$ is then obviously tomorrow or the next period. Another typical value of time is t_0, which is usually the point in time for which initial values of some process are given. Similarly, T denotes a future point in time where, for example, the planning horizon ends or for which some boundary values are given. In most cases, time τ refers to some future point in time in the sense of $\tau \geq t$.

Given these definitions, how should one present generic transition equations, in most examples above budget constraints? Should one use time t as argument or time τ? In two-period models, it is most natural to use t for today and $t+1$ for tomorrow. This is the case in the two-period models of ch. 2, for example, in the budget constraints (2.1.2) and (2.1.3).

This choice becomes less obvious in multi-period models of ch. 3. When the first transition equation appears in (3.3.1) and the first dynamic (in contrast to intertemporal) budget constraint in (3.4.1), one can make arguments both in favour of t or τ as time argument. Using τ would be most general: The transition equation is valid for all periods in time, hence one should use τ. On the other hand, when the transition equation is valid for t as today and as we know that tomorrow will be "today" tomorrow (or: "today turned into yesterday tomorrow - morgen ist heute schon gestern") and that any future point in time will be today at some point, we could use t as well.

In most cases, the choice of t or τ as time argument is opportunistic: When a maximization method is used where the most natural representation of budget constraints is in t, we will use t. This is the case for the transition equation (3.3.1) where the maximization problem is solved by using a Bellman equation. Using t is also most natural when presenting the setup of a model as in ch. 3.6.

If by contrast, the explicit use of many time periods is required in a maximization problem (like when using the Lagrangian with an intertemporal budget constraint as in (3.1.3) or with a sequence of dynamic budget constraints in (3.7.2)), budget constraints are expressed with time τ as argument. Using τ as argument in transition equations is also more appropriate for Hamiltonian problems where the current value Hamiltonian is used - as throughout this book. See, for example, the budget constraint (5.1.2) in ch. 5.1.

In the more formal ch. 4 on differential equations, it is also most natural (as this is the case in basically all textbooks on differential equations) to represent all differential equation with t as argument. When we solve differential equations, we need boundary conditions. When we use initial conditions, they will be given by t_0. If we use a terminal condition as boundary condition, we use $T > t$ to denote some future point in time. (We could do without t_0, use τ as argument and solve for $t_0 \leq \tau \leq T$. While this would be more consistent with the rest of the book, it would be less comparable to more specialized books on differential equations. As we believe that the "inconsistency" is not too strong, we stick to the more standard mathematical notation in ch. 4).

Here is now a summary of points in time and generic time

t	a point in time t standing for *t*oday (discrete and continuous time)
t	generic time (for differential equations in ch. 4 and for model presentations)
$t+1$	tomorrow (discrete time)
τ	some future point in time, $\tau \geq t$ (discrete and continuous time)
τ	generic time (for differential equations in some maximization problems - e.g. when Hamiltonians are used)
t_0	a point in time, usually in the past (differential equations)
T	a point in time, usually in the future (differential equations and terminal point for planning horizon)
0	today when t is normalized to zero (only in the examples of ch. 5.5.1)

More on notation

The notation is as homogeneous as possible. Here are the general rules but some exceptions are possible.

- Variables in capital, like capital K or consumption C denote aggregate quantities, lower-case letters pertain to the household level
- A function $f(.)$ is always presented by using parentheses $(.)$, where the parentheses give the arguments of functions. Brackets [.] always denote a multiplication operation
- A variable x_t denotes the value of x in period t in a discrete time setup. A variable $x(t)$ denotes its value at a point in time t in a continuous time setup. We will stick to this distinction in this textbook both in deterministic and stochastic setups. (Note that the mathematical literature on stochastic continuous time processes, i.e. what is treated here in part IV, uses x_t to express the value of x at a point in time t.)

- A dot indicates a time derivative, $\dot{x}(t) \equiv dx/dt$.
- A derivative of a function $f(x)$ where x is a scalar and not a vector is abbreviated by $f'(x) \equiv df(x)/dx$. Partial derivatives of a function $f(x, y)$ are denoted by for example, $f_x(x, y) = f_x(.) = f_x \equiv \partial f(x, y)/\partial x$.

Variables

Here is a list of variables and abbreviations which shows that some variables have multi-tasking abilities, i.e. they have multiple meanings

- Greek letters

α	output elasticity of capital
$\beta = 1/(1+\rho)$	discount factor in discrete time
γ	preference parameter on first period utility in two-period models
δ	depreciation rate
θ	share of wealth held in the risky asset
θ_i	share of wealth held in asset i
π	instantaneous profits, i.e. profits π_t in period t or $\pi(t)$ at point in time t
ρ	time preference rate, correlation coefficient between random variables
ρ_{ij}	correlation coefficient between two random variables i and j
τ	see above on "the concept of time"
χ_i	the share of savings used for buying stock i
$\looparrowleft \divideontimes$	misprint
Φ	adjustment cost function

- Latin letters

$\{c_\tau\}$	the time path of c from t to infinity, i.e. for all $\tau \geq t$, $\{c_\tau\} \equiv \{c_t, c_{t+1}, ...\}$
$\{c(\tau)\}$	the time path of c from t to infinity, i.e. for all $\tau \geq t$
CVF	change of variable formula
e	expenditure $e = pc$, effort, exponential function
$f(x+y)$	a function $f(.)$ with argument $x+y$
$f[x+y]$	some variable f times $x+y$
$q(t)$	Poisson process
r	interest rate
RBC	real business cycle

ODE	ordinary differential equation
RV	random variable
SDE	stochastic differential equation
t, T, t_0	see above on "the concept of time"
TVC	transversality condition
TFP	total factor productivity
u	instantaneous utility (see also π)
w_t, w_t^L	wage rate in period t; factor reward for labour. w_t^L is used to stress difference to w_t^K
w_t^K	factor reward for capital in period t
x^*	fixpoint of a difference or differential equation (system)
$z(t)$	Brownian motion, Wiener process

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Index

Part V

Solutions manual

Appendix A

Introduction to solutions to exercises

This solutions manual presents solutions to selected exercises. There are not solutions to all exercises simply as not all of the existing solutions are nicely typed, edited and checked for errors at this point. There are solutions to the following exercises.

Chapter 1 – no exercises
Chapter 2 – Exercises 2, 8 and 10
Chapter 3 – Ex. 2, 5 and 7
Chapter 4 – 1, 2, 6, 8 and 9
Chapter 5 – 1, 4, 5 and 8
Chapter 6 – 1
Chapters 7 and 8 – 6
Chapter 9 – 2, 7, 8 and 10
Chapter 10 – 3, 4, 5, 12 and 13
Chapter 11 – 1, 2 and 3

Any book goes through various stages before actually being "born" in its printed version. This is also true for this book. It started with simple notes, went through its adolescence as various informal versions for the internet and became grown-up via the incorporation of serious and thoughtful comments by referees.

I should probably mention that the solutions manual is not as well-tested as the text itself. While I believe that the solutions are correct, the ideas behind them are not explained in an as elaborated way as in the main text – after all, it is a manual. I am nevertheless sure that it will be very useful for users of this book. As always, any comments are welcome.

The main text has been written word for word by the author of the book. The solutions of the exercises were all suggested by the author. A large group of collaborators have contributed, edited and refined them. They include Ken Sennewald, Christopher Kops and Michael Lamprecht. I am highly grateful to them for their support.

Appendix B

Solution to exercises of chapter 2

B.1 Solving by substitution – ex. 2

From the main text (see equations (2.2.1), (2.1.2) and (2.1.3)) we already know the following equations

$$\text{objective function:} \qquad U_t = \gamma \ln c_t + (1-\gamma) \ln c_{t+1} \tag{B.1.1}$$

$$\text{budget constraint in } t\text{:} \qquad c_t = w_t - s_t \tag{B.1.2}$$

$$\text{budget constraint in } t+1\text{:} \qquad c_{t+1} = w_{t+1} + (1 + r_{t+1}) s_t. \tag{B.1.3}$$

Now the principle is to transform this optimization problem with constraints into one without constraints. We therefore insert the budget constraints (B.1.2) and (B.1.3) into our objective function (B.1.1). The unconstrained maximization problem then reads $\max U_t$ by choosing s_t, where

$$U_t = \gamma \ln(w_t - s_t) + (1-\gamma) \ln(w_{t+1} + (1 + r_{t+1}) s_t).$$

The derivative with respect to savings s_t is given by

$$\frac{\gamma}{w_t - s_t} = (1 + r_{t+1}) \frac{1-\gamma}{w_{t+1} + (1 + r_{t+1}) s_t},$$

which represents the first-order condition. When this is solved for savings s_t, we obtain

$$\frac{w_{t+1}}{1 + r_{t+1}} + s_t = (w_t - s_t) \frac{1-\gamma}{\gamma} \Leftrightarrow \frac{w_{t+1}}{1 + r_{t+1}} = w_t \frac{1-\gamma}{\gamma} - s_t \frac{1}{\gamma} \Leftrightarrow$$

$$s_t = (1-\gamma) w_t - \gamma \frac{w_{t+1}}{1 + r_{t+1}} = w_t - \gamma W_t,$$

where W_t is life-time income as defined after (2.2.3).

To show that this is the same result as in (2.2.4) and (2.2.5) we compute first- and second-period consumption and find

$$c_t = w_t - s_t = \gamma W_t$$

$$c_{t+1} = w_{t+1} + (1 + r_{t+1}) s_t = w_{t+1} + (1 + r_{t+1})(w_t - \gamma W_t) = (1 + r_{t+1})(1-\gamma) W_t,$$

where the last equation used again the definition (2.2.3) of life-time income W_t.

B.2 General equilibrium – ex. 8

Exercise 8 a)

- **The firms**

The technology is given by

$$Y_t = K_t^\alpha L^{1-\alpha}. \tag{B.2.1}$$

As in section 2.4, we choose Y_t as our numéraire good and normalize its price to unity, $p_t = 1$. With this normalization, profits are given by $\pi_t = Y_t - w_t^K K - w_t^L L$. Sticking to section 2.4, we let firms act under perfect competition and get the following first-order conditions,

$$w_t^K = \frac{\partial Y_t}{\partial K_t} \overset{(B.2.1)}{=} \alpha K_t^{\alpha-1} L^{1-\alpha}, \tag{B.2.2}$$

$$w_t^L = \frac{\partial Y_t}{\partial L} \overset{(B.2.1)}{=} (1-\alpha) K_t^\alpha L^{-\alpha}. \tag{B.2.3}$$

- **Households**

The utility function is given by $U_t = \ln c_t^y + \beta \ln c_{t+1}^o$. It is maximized subject to the intertemporal budget constraint

$$w_t = c_t^y + (1 + r_{t+1})^{-1} c_{t+1}^o.$$

Note that households work in the first period, but retire in the second. Hence, there is labour income only in the first period on the left-hand side and savings of the first period are used to finance consumption in the second period. Given that the present value of lifetime wage income is w_t, we can compute individual consumption expenditure and savings,

$$c_t^y = \frac{1}{1+\beta} w_t, \quad c_{t+1}^o = \frac{\beta}{1+\beta}(1 + r_{t+1}) w_t,$$

$$s_t = w_t - c_t^y = \frac{\beta}{1+\beta} w_t \tag{B.2.4}$$

Assuming that all individuals within a generation are identical, then aggregate consumption in t is given by

$$C_t = Lc_t^y + Lc_t^o = \left(\frac{1}{1+\beta} w_t + \frac{\beta}{1+\beta}(1 + r_t) w_{t-1} \right) L.$$

- **Goods market equilibrium**

Sticking to subsection 2.4.3, the goods market equilibrium requires that supply equals demand, $Y_t = C_t + I_t$, where demand is given by consumption plus gross investment. The next period's capital stock is then given by $K_{t+1} = I_t + (1-\delta) K_t$, where δ is the depreciation rate. These two equations imply

$$Y_t + (1-\delta) K_t = C_t + K_{t+1}, \tag{B.2.5}$$

where we simply replaced gross investment by the goods market equilibrium.

- **Reduced form**

Providing that the technology used by firms to produce Y_t has constant returns to scale, we obtain (see equations (2.4.3) and (2.4.4)) from Euler's theorem that

$$Y_t = \frac{\partial Y_t}{\partial K_t} K_t + \frac{\partial Y_t}{\partial L_t} L_t.$$

and this yields with equations (B.2.2) and (B.2.3) the following relation $Y_t = w_t^K K + w_t^L L$. Using this for (B.2.5) and splitting aggregate consumption into consumption of the young and consumption of the old gives

$$w_t^K K_t + w_t^L L + (1-\delta) K_t = C_t^y + C_t^o + K_{t+1}.$$

Defining the interest rate r_t as the difference between factor rewards for capital w_t^K and the depreciation rate δ, $r_t \equiv w_t^K - \delta$, we find

$$r_t K_t + w_t^L L + K_t = C_t^y + C_t^o + K_{t+1}.$$

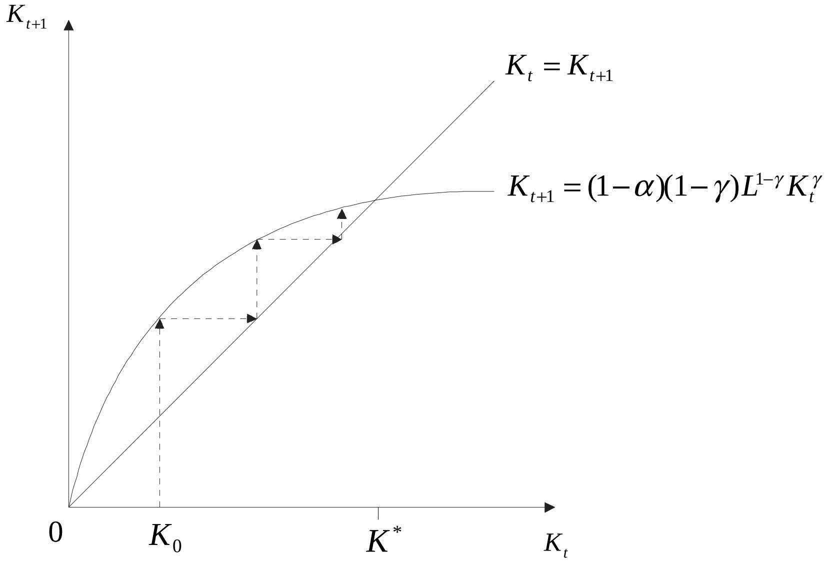
As the old consume the capital stock plus interest $C_t^o = (1+r_t) K_t$, we obtain

$$K_{t+1} = w_t^L L - C_t^y = \left(w_t^L - c_t^y\right) L \overset{(B.2.4)}{=} \frac{\beta}{1+\beta} w_t^L L$$

$$\overset{(B.2.3)}{\Leftrightarrow} K_{t+1} = \frac{\beta}{1+\beta}(1-\alpha) L^{1-\alpha} K_t^{\alpha} \tag{B.2.6}$$

Exercise 8 b)

The phase diagram depicts K_t on the horizontal and K_{t+1} on the vertical axis.



Exercise 8 c)

- **Steady state capital stock**

In the steady state, the capital stock is constant, $K_t = K_{t+1} = K^*$. Inserting in (B.2.6) yields

$$K^* = \frac{\beta}{1+\beta}(1-\alpha)L^{1-\alpha}(K^*)^\alpha$$

$$\Leftrightarrow (K^*)^{1-\alpha} = \frac{\beta}{1+\beta}(1-\alpha)L^{1-\alpha} \Leftrightarrow K^* = \left[\frac{\beta}{1+\beta}(1-\alpha)\right]^{\frac{1}{1-\alpha}} L.$$

- **Steady state consumption level**

Substituting $K_{t+1} = K_t = K^*$ into (B.2.5) gives

$$Y + (1-\delta)K^* = C + K^* \Leftrightarrow C^* = Y(K^*) - \delta K^*.$$

B.3 Difference equations – ex. 10

Consider the following linear difference equation,

$$y_{t+1} = ay_t + b, \quad a < 0 < b.$$

Exercise 10 a)

A fixpoint y^* satisfies $y^* = ay^* + b$, hence it is given by $y^* = \frac{b}{1-a}$, where clearly $a \neq 0$.

Exercise 10 b)

Starting with y_0, we obtain via solving by substitution

$$\begin{aligned} y_1 &= ay_0 + b, \\ y_2 &= ay_1 + b = a^2 y_0 + ab + b, \\ y_3 &= ay_2 + b = a^3 y_0 + \left(a^2 + a + 1\right) b, \\ &\vdots \\ y_t &= a^t y_0 + b \sum_{i=0}^{t-1} a^i. \end{aligned}$$

Consequently, the limit for $t \to \infty$ is computed via

$$\lim_{t\to\infty} y_t = y_0 \lim_{t\to\infty} a^t + b \lim_{t\to\infty} \sum\nolimits_{i=0}^{t-1} a^i = y_0 \lim_{t\to\infty} a^t + \frac{b}{1-a},$$

where we had to assume $|a| < 1$ in order to use the limit of the geometric series. Nevertheless, we have to distinguish between the following three cases:

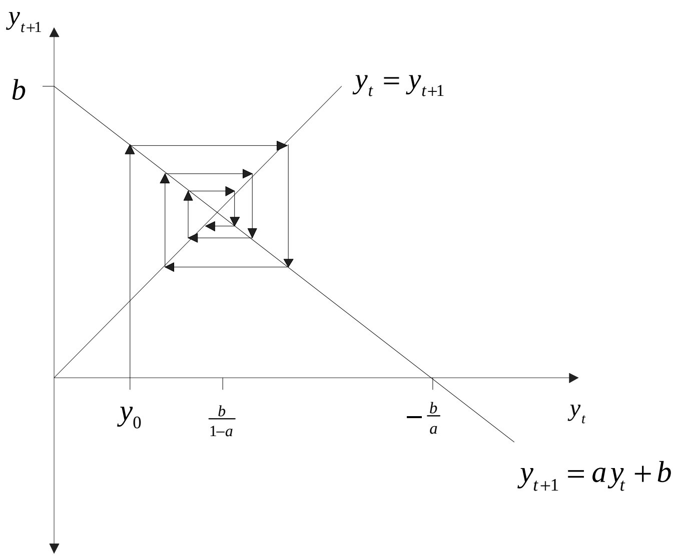
1. For $-1 < a < 0$ we have $\lim_{t\to\infty} y_t = 0 + \frac{b}{1-a} = \frac{b}{1-a}$.
2. For $a = -1$, the series does not converge.
3. For $a < -1$ the sequence is unbounded and alternating. If $y_0(1-a) > b$, the subsequence $\{y_{2k}\}$ diverges to ∞, whereas the subsequence $\{y_{2k+1}\}$ diverges to $-\infty$. If $y_0(1-a) < b$ the subsequence $\{y_{2k}\}$ diverges to $-\infty$ and the subsequence $\{y_{2k+1}\}$ diverges to ∞.

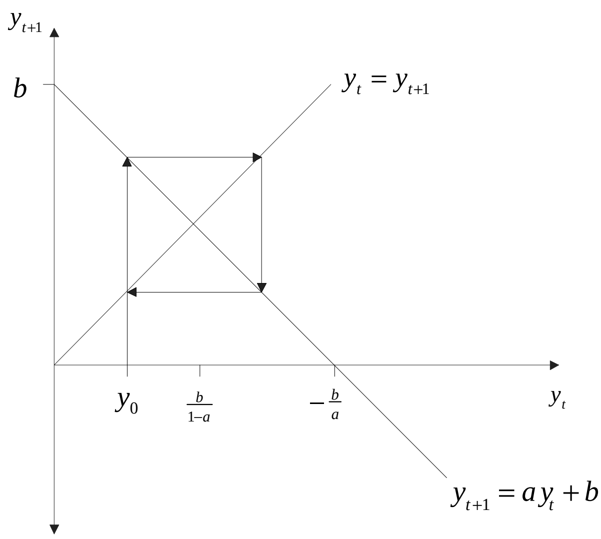
Hence the fixpoint y^* is only stable for $-1 < a < 0$.

Exercise 10 c)

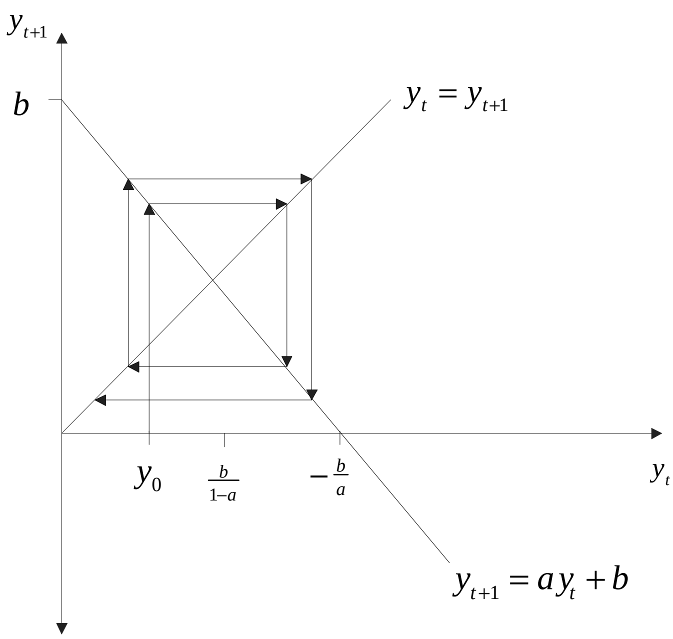
We need to distinguish between three cases here as well.

(i) for $-1 < a < 0$ (left panel) and (ii) for $a = -1$ (right panel)





(iii) for $a < -1$



Appendix C

Solution to exercises of chapter 3

C.1 The envelope theorem II – ex. 2

Exercise 2 a)

The Bellman equation reads $V(a_t) = \max_{c_t}\{u(c_t) + \beta V(a_{t+1})\}$ and the derivative with respect to a_t of the maximized Bellman equation reads

$$V'(a_t) = u'(c_t)\frac{dc_t}{da_t} + \beta V'(a_{t+1})\frac{da_{t+1}}{da_t}.$$

Using the budget constraint $a_{t+1} = (1 + r_t)(a_t + w_t - c_t)$, we obtain

$$V'(a_t) = u'(c_t)\frac{dc_t}{da_t} + \beta V'(a_{t+1})(1 + r_t)(1 - \frac{dc_t}{da_t}).$$

As we did not employ the envelope theorem, we can now use the first order condition in order to remove the derivatives of the control variable with respect to the state variable. This yields

$$\begin{aligned} V'(a_t) &= \beta V'(a_{t+1})(1 + r_t)\frac{dc_t}{da_t} + \beta V'(a_{t+1})(1 + r_t)(1 - \frac{dc_t}{da_t}) \\ &= \beta V'(a_{t+1})(1 + r_t) = \beta V'(a_{t+1})\frac{\partial a_{t+1}}{\partial a_t}. \end{aligned}$$

Exercise 2 b)

Starting out with the Bellman equation $V(M_t) = \max_{e_t}\{-e_t + \beta V(M_{t+1})\}$, the derivative with respect to M_t of the maximized Bellman equation reads

$$V'(M_t) = -\frac{\partial e_t}{\partial M_t} + \beta V'(M_{t+1})\frac{\partial M_{t+1}}{\partial M_t}.$$

Not having used the envelope theorem, we see the derivatives $\frac{\partial e_t}{\partial M_t}$ of the control variable in this equation. Using the constraint $M_{t+1} = f(e_t) + M_t$, we have

$$V'(M_t) = -\frac{\partial e_t}{\partial M_t} + \beta V'(M_{t+1}) \left(f'(e_t) \frac{\partial e_t}{\partial M_t} + 1 \right).$$

But, as the first order condition is given by $1 = \beta V'(M_{t+1}) f'(e_t)$, the expression above simplifies to $V'(M_t) = \beta V'(M_{t+1})$ which, due to its intertemporal nature, implies

$$V'(M_{t+1}) = \beta V'(M_{t+2}).$$

C.2 The standard saving problem – ex. 5

Exercise 5 a)

We start from the following two fundamentals of our problem.

Utility function:	$U_t = \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau)$	(C.2.1)
Budget constraint:	$k_{t+1} = (1 + r_t) k_t + w_t^L - c_t$	(C.2.2)

- DP 1: Bellman equation and first-order condition

The utility function (C.2.1) can be rewritten as $U_t = u(c_t) + \beta U_{t+1}$ and by applying Bellman's idea we obtain

$$V(k_t) \equiv \max_{c_t} U_t = \max_{c_t} \{u(c_t) + \beta V(k_{t+1})\} \tag{C.2.3}$$

subject to (C.2.2). The first-order condition of maximizing with respect to c_t reads

$$\frac{d}{dc_t} u(c_t) + \beta \frac{d}{dc_t} V(k_{t+1}) = u'(c_t) + \beta V'(k_{t+1}) \frac{dk_{t+1}}{dc_t} = 0$$

as $k_{t+1} = k_{t+1}(c_t)$ by the budget constraint (C.2.2). Since $dk_{t+1}/dc_t = -1$, we find as first order condition

$$u'(c_t) = \beta V'(k_{t+1}). \tag{C.2.4}$$

- DP 2: Evolution of the costate variable

Using the envelope theorem, the derivative of Bellman's equation (C.2.3) reads

$$V'(k_t) = \beta V'(k_{t+1}) \frac{dk_{t+1}}{dk_t} \overset{(C.2.2)}{=} \beta (1 + r_t) V'(k_{t+1}). \tag{C.2.5}$$

- DP 3: Inserting first-order conditions

Substituting the first order condition for t and $t+1$ into equation (C.2.5) yields $\beta^{-1} u'(c_{t+1}) = \beta(1+r_t) \beta^{-1} u'(c_t)$. Lagging this equation by one period finally gives $u'(c_t) = \beta(1 + r_{t+1}) u'(c_{t+1})$.

Exercise 5 b)

Now we use the Lagrange approach with an infinite sequence of constraints. Hence we have

$$\mathcal{L} = \Sigma_{\tau=t}^{\infty} \left\{ \beta^{\tau-t} u(c_\tau) + \lambda_\tau \left[k_{\tau+1} - (1+r_\tau)k_\tau - w_\tau^L + c_\tau \right] \right\}.$$

In order to make the procedure clearer, we rewrite the Lagrangian as

$$\begin{aligned}
\mathcal{L} = & \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) + \Sigma_{\tau=t}^{s-2} \lambda_\tau \left[k_{\tau+1} - (1+r_\tau)k_\tau - w_\tau^L + c_\tau \right] \\
& + \lambda_{s-1} \left[k_s - (1+r_{s-1})k_{s-1} - w_{s-1}^L + c_{s-1} \right] \\
& + \lambda_s \left[k_{s+1} - (1+r_s)k_s - w_s^L + c_s \right] \\
& + \Sigma_{\tau=s+1}^{\infty} \lambda_\tau \left[k_{\tau+1} - (1+r_\tau)k_\tau - w_\tau^L + c_\tau \right].
\end{aligned}$$

This Lagrangian clearly shows the infinite number of constraints and multipliers. It also allows us to write the first-order conditions for time s as

$$\begin{aligned}
\mathcal{L}_{c_s} &= \beta^{s-t} u'(c_s) + \lambda_s = 0 \Leftrightarrow \lambda_s = -u'(c_s)\beta^{s-t}, \\
\mathcal{L}_{k_s} &= \lambda_{s-1} - \lambda_s(1+r_s) = 0 \Leftrightarrow \frac{\lambda_{s-1}}{\lambda_s} = 1 + r_s, \\
\mathcal{L}_{\lambda_s} &= 0.
\end{aligned}$$

Combining the first and second first-order conditions gives

$$\frac{-u'(c_{s-1})\beta^{s-1-t}}{-u'(c_s)\beta^{s-t}} = 1 + r_s.$$

Hence, by lagging this result one period, we obtain

$$u'(c_s) = \beta(1 + r_{s+1})u'(c_{s+1}).$$

Exercise 5 c)

We can also first derive an intertemporal budget constraint. To this end, we rewrite the budget constraint (C.2.2) as $k_t = \frac{k_{t+1}+c_t-w_t^L}{1+r_t}$ or as $k_{t+i} = \frac{k_{t+i+1}+c_{t+i}-w_{t+i}^L}{1+r_{t+i}}$. Substituting sufficiently often yields

$$\begin{aligned}
k_t &= \frac{\frac{k_{t+2}+c_{t+1}-w_{t+1}^L}{1+r_{t+1}} + c_t - w_t^L}{1+r_t} = \frac{k_{t+2} + c_{t+1} - w_{t+1}^L}{(1+r_{t+1})(1+r_t)} + \frac{c_t - w_t^L}{1+r_t} \\
&= \frac{\frac{k_{t+3}+c_{t+2}-w_{t+2}^L}{1+r_{t+2}} + c_{t+1} - w_{t+1}^L}{(1+r_{t+1})(1+r_t)} + \frac{c_t - w_t^L}{1+r_t} \\
&= \frac{k_{t+3} + c_{t+2} - w_{t+2}^L}{(1+r_{t+2})(1+r_{t+1})(1+r_t)} + \frac{c_{t+1} - w_{t+1}^L}{(1+r_{t+1})(1+r_t)} + \frac{c_t - w_t^L}{1+r_t} \\
&= \cdots = \lim_{i\to\infty} \frac{k_{t+i}}{(1+r_{t+i-1})\dots(1+r_{t+1})(1+r_t)} + \sum_{i=0}^{\infty} \frac{c_{t+i} - w_{t+i}^L}{(1+r_{t+i-1})\dots(1+r_{t+1})(1+r_t)}
\end{aligned}$$

and thus we obtain

$$k_t = \lim_{i\to\infty} \frac{k_{t+i}}{\prod_{s=0}^{i-1}(1+r_{t+s})} + \sum_{i=0}^{\infty} \frac{c_{t+i} - w_{t+i}^L}{\prod_{s=0}^{i}(1+r_{t+s})}.$$

Using the no-Ponzi game condition (see ch. 3.5.2), we assume that the limit is zero. Then, by substituting $\tau = t + i$, we obtain

$$k_t = \Sigma_{\tau=t}^{\infty} \frac{c_\tau - w_\tau^L}{\prod_{s=0}^{\tau-t}(1+r_{t+s})} = \Sigma_{\tau=t}^{\infty} \frac{c_\tau}{\prod_{s=0}^{\tau-t}(1+r_{t+s})} - \Sigma_{\tau=t}^{\infty} \frac{w_\tau^L}{\prod_{s=0}^{\tau-t}(1+r_{t+s})}$$

and the desired intertemporal budget constraint with a constant interest rate reads

$$\Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t+1)}c_\tau = k_t + \Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t+1)}w_\tau^L.$$

Hence, the Lagrangian reads

$$\mathcal{L} = \Sigma_{\tau=t}^{\infty}\beta^{\tau-t}u(c_\tau) + \lambda\left[\Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t+1)}c_\tau - k_t - \Sigma_{\tau=t}^{\infty}(1+r)^{-(\tau-t+1)}w_\tau^L\right]$$

where λ is the Lagrange multiplier.

First-order conditions are

$$\mathcal{L}_{c_\tau} = \beta^{\tau-t}u'(c_\tau) + \lambda(1+r)^{-(\tau-t+1)} = 0,$$
$$\mathcal{L}_\lambda = 0.$$

The first-order condition for period $\tau = t$ and for period $\tau = t + 1$ read

$$u'(c_t) + \lambda(1+r)^{-1} = 0,$$
$$\beta u'(c_{t+1}) + \lambda(1+r)^{-2} = 0,$$

and dividing them gives

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = (1+r) \quad \Rightarrow \quad u'(c_t) = \beta(1+r)u'(c_{t+1}).$$

C.3 A benevolent central planner – ex. 7

Exercise 7 a)

Our starting point consists of the following equations,

Social welfare function:	$U_t = \Sigma_{\tau=t}^{\infty}\beta^{\tau-t}u(C_\tau)$	(C.3.1)
Resource constraint:	$K_{t+1} - K_t = Y(K_t, L_t) - \delta K_t - C_t.$	(C.3.2)

- DP 1: Bellman equation and first-order condition

The value function for this problem reads $V(K_t) \equiv \max_{\{C_\tau\}} U_t$ s.t. (C.3.2) and the Bellman equation reads $V(K_t) = \max_{C_t}[u(C_t) + \beta V(K_{t+1})]$. The first-order condition of maximizing with respect to C_t reads

$$u'(C_t) + \beta V'(K_{t+1}) \frac{dK_{t+1}}{dC_t} = 0$$

as $K_{t+1} = K_{t+1}(C_t)$ by the resource constraint (C.3.2). From the budget constraint we see that $dK_{t+1}/dC_t = -1$, and therefore the first-order condition is

$$u'(C_t) = \beta V'(K_{t+1}). \tag{C.3.3}$$

- DP 2: Evolution of the costate variable

Since $C_t = C_t(K_{t+1})$ by first-order condition (C.3.3) and resource constraint (C.3.2), we get $V(K_{t+1}) = u(C_t(K_{t+1})) + \beta V(K_{t+2})$. Employing the envelope theorem, we have

$$V'(K_{t+1}) = \beta V'(K_{t+2}) \frac{dK_{t+2}}{dK_{t+1}}.$$

Employing the resource constraint $K_{t+2} = Y(K_{t+1}, L_{t+1}) + K_{t+1} - \delta K_{t+1} - C_{t+1}$, this is equal to

$$V'(K_{t+1}) = \beta V'(K_{t+2}) \left(\frac{\partial Y}{\partial K_{t+1}} + 1 - \delta \right). \tag{C.3.4}$$

- DP 3: Inserting first-order conditions

From (C.3.3) we know that $u'(C_{t+1}) = \beta V'(K_{t+2})$. Hence, equation (C.3.4) reads $V'(K_{t+1}) = u'(C_{t+1}) \left(\frac{\partial Y}{\partial K_{t+1}} + 1 - \delta \right)$. Using this in (C.3.3) again, we find $u'(C_t) = \beta u'(C_{t+1}) \left(\frac{\partial Y}{\partial K_{t+1}} + 1 - \delta \right)$ and thus

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{1}{\frac{1}{1 + \frac{\partial Y}{\partial K_{t+1}} - \delta}}. \tag{C.3.5}$$

Exercise 7 b)

The main difference between exercise 5 and 7 of chapter 3 is that the first one considers the optimal saving problem of a household in a decentralized economy. This exercise considers optimal saving of a central planner.

When we compare the Euler equations of the two setups, we find that they are the same if $\frac{\partial Y}{\partial K_{t+1}} - \delta = r_{t+1}$, i.e. if the marginal productivity of capital corrected for the depreciation rate equals the interest rate individuals face. One condition for the factor allocation to be identical in the planner solution and in the decentralized solution is therefore that the interest rate in the decentralized solution obeys this equation.

Exercise 7 c)

Two standard versions of instantaneous utility functions are the following,

$$\text{Logarithmic utility function:} \qquad u(C_\tau) = \phi \ln C_\tau \tag{C.3.6}$$

$$\text{CES utility function:} \qquad u(C_\tau) = \frac{C_\tau^{1-\sigma} - 1}{1-\sigma} \tag{C.3.7}$$

If we want to substitute the logarithmic utility function (C.3.6) into (C.3.5), we first compute marginal utility, which gives $u'(C_\tau) = \phi C_\tau^{-1}$. We then obtain

$$C_{t+1} = \left(\frac{\beta}{\frac{1}{1+\frac{\partial Y}{\partial K_{t+1}}-\delta}} \right) C_t$$

Let us now use the CES utility function (C.3.7) instead for (C.3.5). Computing marginal utility gives $u'(C_\tau) = C_\tau^{-\sigma}$ and we obtain a linear difference equation in consumption which reads

$$C_{t+1} = \left(\frac{\beta}{\frac{1}{1+\frac{\partial Y}{\partial K_{t+1}}-\delta}} \right)^{\frac{1}{\sigma}} C_t.$$

Appendix D

Solution to exercises of chapter 4

D.1 Phase diagram I – ex. 1

Exercise 1 a)

We first construct zero-motion lines and find

$$\dot{x}_1 = 0 \Leftrightarrow f(x_1, x_2) = 0, \quad \dot{x}_2 = 0 \Leftrightarrow g(x_1, x_2) = 0$$

The slope of the zero-motion lines follows from the properties of $f(.)$ and $g(.)$ as stated in the problem,

$$\left.\frac{dx_2}{dx_1}\right|_{f(x_1,x_2)=0} < 0, \quad \left.\frac{dx_2}{dx_1}\right|_{g(x_1,x_2)=0} > 0.$$

The zero-motion lines can be linear or non-linear. The following figure draws the linear case.

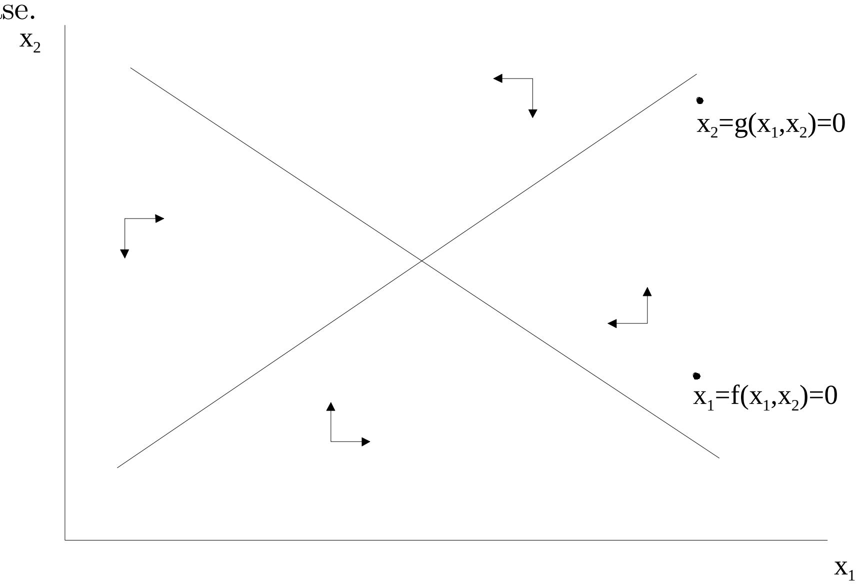


Figure D.1.1 *Phase diagram I*

Since $f_{x_1} < 0$, $g_{x_2} < 0$ are given and the implicit function theorem (see equation (2.3.3)) states

$$\frac{dx_2}{dx_1} = -\frac{\partial h(.)/\partial x_1}{\partial h(.)/\partial x_2}$$

we have $f_{x_2} < 0$ and $g_{x_1} > 0$. Hence, pairs of arrows are explained by

- $f_{x_1} < 0$: Starting from the $\dot{x}_1 = f(x_1, x_2) = 0$ line, a small movement to the right will decrease the value of $\dot{x}_1$. Hence, for any points to the right of $f(x_1, x_2) = 0$, x_1 is decreasing, i.e. $\dot{x}_1 < 0$.

- $f_{x_2} < 0$: Starting from the $\dot{x}_1 = f(x_1, x_2) = 0$ line, a small movement downwards will increase the value of $\dot{x}_1$. Hence, for any points below $f(x_1, x_2) = 0$, x_1 is increasing, i.e. $\dot{x}_1 > 0$.

- $g_{x_2} < 0$: Starting from the $\dot{x}_2 = g(x_1, x_2) = 0$ line, a small movement upwards will decrease the value of $\dot{x}_2$. Hence, for any points above $g(x_1, x_2) = 0$, x_2 is decreasing, i.e. $\dot{x}_2 < 0$.

- $g_{x_1} > 0$: Starting from the $\dot{x}_2 = g(x_1, x_2) = 0$ line, a small movement to the right will increase the value of $\dot{x}_2$. Hence, for any points to the right of $g(x_1, x_2) = 0$, x_2 is increasing, i.e. $\dot{x}_2 > 0$.

Exercise 1 b)

A phase diagram analysis allows to identify a saddle point. If no saddle point can be identified, it is generally not possible to distinguish between a node, focus or center. If the given differential equation system was linear, we could use the approach in section 4.5 via eigenvalues.

D.2 Phase diagram II – ex. 2

Exercise 2 a)

Zero motion lines are given by $\dot{x} = 0 \Leftrightarrow y = a/x$ and $\dot{y} = 0 \Leftrightarrow y = b$. The partial derivatives indicating the slope are $\frac{\partial \dot{x}}{\partial x} = y > 0$, $\frac{\partial \dot{x}}{\partial x} = y > 0$, $\frac{\partial \dot{y}}{\partial y} = 1 > 0$ and $-\frac{\partial \dot{y}}{\partial y} = -1 < 0$. Depending on the sign of b, we get the following two phase diagrams.

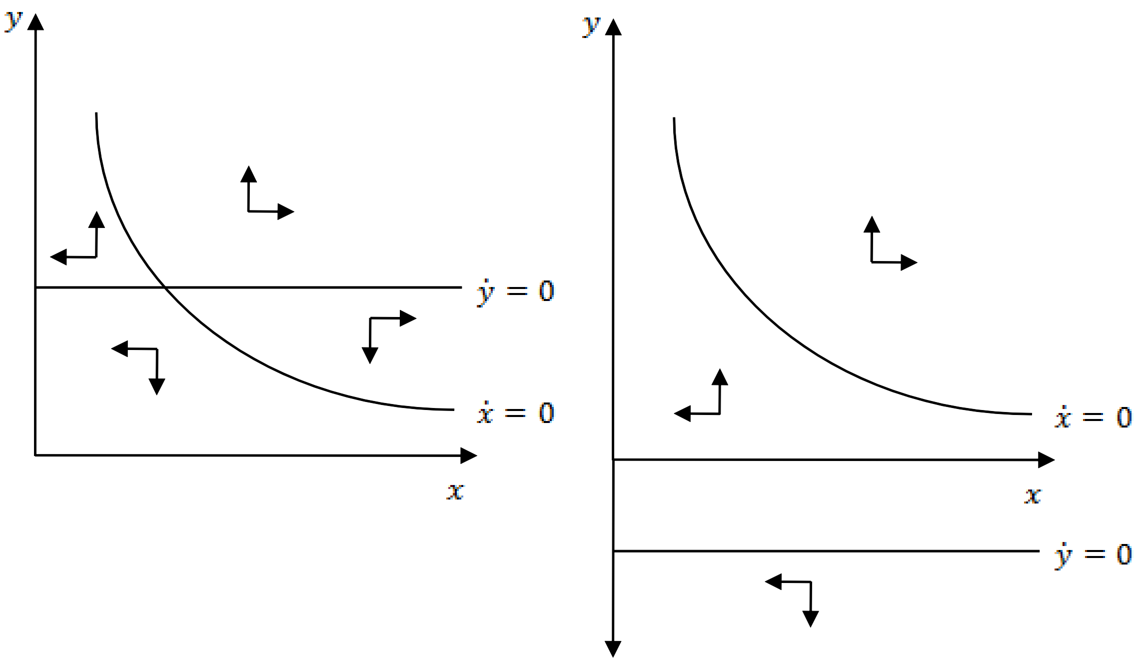


Figure D.2.1 *Phase diagram II*

Exercise 2 b)

Generally speaking, a phase diagram analysis allows to identify a saddle point. If no saddle point can be identified, it is generally not possible to distinguish between a node, focus or center. If the given differential equation system was linear, we could use the approach in section (4.5) via eigenvalues.

The left panel shows the case of a saddle path. The right panel does not display a fixpoint.

Exercise 2 c)

Let us consider the differential equation system

$$\begin{aligned} \dot{x}(\tau) &= x(\tau)y(\tau) - a \\ \dot{y}(\tau) &= y(\tau) - b \end{aligned} \tag{D.2.1}$$

with $\tau \geq t_0$, $t_0 \geq 0$. The initial value of the system is $(x(t_0), y(t_0)) \equiv (x_0, y_0)$. The second element of our system, describes the evolution of the function y in time. Because y evolves independently of x and the differential equation $\dot{y}(\tau) = y(\tau) - b$ is linear, we can use the backward solution (4.3.7) to get the solution

$$\tilde{y}(t) = y_0 e^{t-t_0} + b[1 - e^{t-t_0}] = (y_0 - b)e^{t-t_0} + b \tag{D.2.2}$$

with $t \in [t_0, \infty)$.

Now we can use the solution (D.2.2) for the first element of the system (D.2.1) and get

$$\dot{x}(\tau) = \tilde{y}(\tau)x(\tau) - a = \left((y_0 - b)e^{\tau - t_0} + b\right) x(\tau) - a \tag{D.2.3}$$

with $\tau \geq t_0$. Therefore the solution $\tilde{x} : [t_0, \infty) \to \mathbb{R}$ is described by a linear differential equation (D.2.3). We can use again the backward solution (4.3.7) to get the function $\tilde{x}$.

Before we apply the backward solution, we determine the integral

$$A(\tau, t) = \int_\tau^t \tilde{y}(u)du = \int_\tau^t \left[(y_0 - b)e^{u-t_0} + b\right] du = (y_0 - b)[e^{t-t_0} - e^{\tau - t_0}] + b(t - \tau)$$

with $\tau, t \in [t_0, \infty)$ and $\tau \leq t$. If we insert this result in the backward solution of the differential equation (D.2.3), we obtain the solution

$$\tilde{x}(t) = x_0 e^{A(t_0,t)} - a \int_{t_0}^t e^{A(\tau,t)} d\tau \tag{D.2.4}$$

or more explicitly

$$\begin{aligned} \tilde{x}(t) &= x_0 \exp\left\{(y_0 - b)\left(e^{t-t_0} - 1\right) + b(t - t_0)\right\} \\ &\quad -a \int_{t_0}^t \exp\left\{(y_0 - b)\left(e^{t-\tau} - 1\right) + b(t - \tau)\right\} d\tau. \end{aligned}$$

Summarizing, we have found the solution (D.2.4) and (D.2.2) of the differential equation system (D.2.1) with initial value (x_0, y_0).

D.3 Solving linear differential equations – ex. 6

We start from the differential equation $\alpha\dot{y}(t) + \beta y(t) = \gamma$, with the boundary condition $y(s) = 17$. Assuming $\alpha \neq 0$ and dividing by α yields

$$\dot{y}(t) = -\frac{\beta}{\alpha}y(t) + \frac{\gamma}{\alpha} \Leftrightarrow \dot{y}(t) = ay(t) + b, \tag{D.3.1}$$

where we defined $a \equiv -\beta/\alpha$ and $b \equiv \gamma/\alpha$. (For the case of $\alpha = 0$, we have a trivial problem.) Since α, β, γ are constants, a and b are constants as well. Now from the main text (see equation (4.3.6)) we know that the general solution of the differential equation (D.3.1) reads

$$y\left(t\right) = e^{\int a(t)dt}\left(\tilde{y} + \int e^{-\int a(t)dt} b\left(t\right)dt\right),$$

where $\tilde{y}$ is some arbitrary constant.

In order to obtain one particular solution, some value $y(s)$ at some point in time s has to be fixed. Depending on whether s lies in the future $t < s$ or in the past $s < t$, the equation is solved forward or backward.

Exercise 6 a)

Since $t > s$ we use the backward solution. Our initial condition is $y(s) = 17$. Then the solution of (D.3.1) is

$$y(t) = 17e^{\int_s^t a(\tau)d\tau} + \int_s^t e^{\int_\tau^t a(u)du} b(\tau)d\tau, \tag{D.3.2}$$

according to equation (4.3.7) from the main text. But since a, b are constants, we have

$$\begin{aligned} y(t) &= 17e^{(t-s)a} + b\int_s^t e^{(t-\tau)a}d\tau = 17e^{(t-s)a} - \frac{b}{a}\left[e^{(t-\tau)a}\right]_s^t \\ &= 17e^{(t-s)a} - \frac{b}{a}\left[1 - e^{(t-s)a}\right] = e^{(t-s)a}(17 + \frac{b}{a}) - \frac{b}{a} \\ &= e^{(s-t)\frac{\beta}{\alpha}}(17 - \frac{\gamma}{\beta}) + \frac{\gamma}{\beta}, \end{aligned}$$

where we used $a \equiv -\beta/\alpha$ and $b \equiv \gamma/\alpha$.

Exercise 6 b)

Since $t < s$ we use the forward solution. Our terminal condition is $y(s) = 17$. Then the solution of (D.3.1) is

$$y(t) = 17e^{-\int_t^s a(\tau)d\tau} - \int_t^s e^{-\int_t^\tau a(u)du} b(\tau)d\tau, \tag{D.3.3}$$

according to equation (4.3.8) from the main text. But since a, b are constants, we have

$$\begin{aligned} y(t) &= 17e^{-(s-t)a} - b\int_t^s e^{-(\tau-t)a}d\tau = 17e^{(t-s)a} + \frac{b}{a}\left[e^{(t-\tau)a}\right]_t^s \\ &= 17e^{(t-s)a} + \frac{b}{a}\left[e^{(t-s)a} - 1\right] = e^{(t-s)a}(17 + \frac{b}{a}) - \frac{b}{a} \\ &= e^{(s-t)\frac{\beta}{\alpha}}(17 - \frac{\gamma}{\beta}) + \frac{\gamma}{\beta}, \end{aligned}$$

where we replaced again according to $a \equiv -\beta/\alpha$ and $b \equiv \gamma/\alpha$.

Exercise 6 c)

Obviously we used the forward solution in case $t < s$ and the backward solution in case $t > s$. However, both solutions coincide as anticipated, since swapping the integral limits went along with changing the algebraic sign of the corresponding integral (compare (D.3.2) to (D.3.3)).

D.4 Derivatives of integrals – ex. 8

Exercise 8 a)

Using Leibniz's rule from equation (4.3.3) yields

$$\frac{d}{dy}\left[\int_a^y f(s)ds\right] = \underbrace{\frac{\partial y}{\partial y}}_{=1} fy) - \underbrace{\frac{\partial a}{\partial y}}_{=0} f(a) + \int_a^y \underbrace{\frac{d}{dy} f(s)}_{=0} ds = f(y)$$

Exercise 8 b)

Here we use Leibniz's rule again and obtain

$$\begin{aligned}\frac{d}{dy}\left[\int_a^y f(y,s)ds\right] &= \underbrace{\frac{\partial y}{\partial y}}_{=1} f(y,y) - \underbrace{\frac{\partial a}{\partial y}}_{=0} f(y,a) + \int_a^y \frac{d}{dy} f(y,s)ds \\ &= f(y,y) + \int_a^y \frac{d}{dy} f(y,s)ds.\end{aligned}$$

Exercise 8 c)

Note that there is a trick in this exercise. Write the integral as $\int_a^b f(z)\,dz$, i.e. just use a variable different from y as variable of integration. Then, using Leibniz's rule, we have

$$\frac{d}{dy}\left[\int_a^b f(z)\,dz\right] = \underbrace{\frac{\partial b}{\partial y}}_{=0} f(b) - \underbrace{\frac{\partial a}{\partial y}}_{=0} f(a) + \int_a^b \underbrace{\frac{d}{dy} f(z)}_{=0} dz = 0.$$

Another way of putting this is $\frac{d}{dy}\left[\int_a^b f(y)\,dy\right] = \frac{d}{dy}\left[F(b) - F(a)\right]$, where $F(.)$ is the indefinite integral of $f(.)$. Then, as $F(b) - F(a)$ is a constant, the derivative with respect to y is obviously zero.

Exercise 8 d)

From equation (4.3.2) in the main text we know the definition of an indefinite integral and thus have

$$\frac{d}{dy}F(y) = \frac{d}{dy}\int f(y)\,dy = f(y),$$

where $F(y)$ is called antiderivative, primitive or indefinite integral of $f(y)$.

Exercise 8 e)

The following proof is similar to the one for equation (4.3.4) in the main text. Consider the product rule

$$(x(t)y(t))' = x'(t)y(t) + x(t)y'(t).$$

Integrating both sides yields

$$\int_a^b (x(t)y(t))'dt = \int_a^b x'(t)y(t)dt + \int_a^b x(t)y'(t)dt,$$

as the integral of a linear combination is the linear combination of the integrals. It follows that

$$\left[\int \frac{d(x(t)y(t))}{dt}dt\right]_a^b = \int_a^b x'(t)y(t)dt + \int_a^b x(t)y'(t)dt$$
$$\Leftrightarrow \int_a^b x'(t)y(t)dt = [x(t)y(t)]_a^b - \int_a^b x(t)y'(t)dt.$$

D.5 Intertemporal and dynamic budget constraints – ex. 9

Exercise 9 a)

We start from the following two budget constraints (BCs),

intertemporal BC: $$E(t) + \dot{A}(t) = r(t)A(t) + I(t), \tag{D.5.1}$$

dynamic BC: $$\int_t^\infty D_r(\tau)E(\tau)d\tau = A(t) + \int_t^\infty D_r(\tau)I(\tau)d\tau, \tag{D.5.2}$$

where D_τ defined as

$$D_r(\tau) \equiv \exp\left[-\int_t^\tau r(s)ds\right]. \tag{D.5.3}$$

We have to show that (D.5.1) implies (D.5.2), if and only if

$$\lim_{T\to\infty} A(T)\exp\left[-\int_t^T (\tau)d\tau\right] = 0. \tag{D.5.4}$$

We start by rewriting (D.5.1) as $\dot{A}(t) = r(t)A(t) + I(t) - E(t)$ and thus recognize that it is a linear differential equation in assets $A(t)$. Its general forward solution by (4.3.8) is given by

$$A(t) = A(T)e^{-\int_t^T r(\tau)d\tau} - \int_t^T \{I(\tau) - E(\tau)\}\, e^{-\int_t^\tau r(s)ds}d\tau.$$

Inserting $D_r(\tau)$ via expression (D.5.3) yields

$$\int_t^T D_r(\tau)I(\tau)d\tau + A(t) = A(T)e^{-\int_t^T r(\tau)d\tau} + \int_t^T D_r(\tau)E(\tau)d\tau,$$

since the integral of a linear combination is the linear combination of the integrals. Letting T go to infinity we obtain

$$\int_t^\infty D_r(\tau)I(\tau)d\tau + A(t) = \lim_{T\to\infty} A(T)e^{-\int_t^T r(\tau)d\tau} + \int_t^\infty D_r(\tau)E(\tau)d\tau$$

and this equation coincides with the dynamic budget constraint (D.5.2) if and only if equation (D.5.4) is valid.

Exercise 9 b)

By defining

$$f(\tau) \equiv D_r(\tau)E(\tau), \quad g(\tau) \equiv D_r(\tau)I(\tau) \tag{D.5.5}$$

the intertemporal budget constraint (D.5.1) simplifies to

$$\int_t^\infty f(\tau)d\tau = A(t) + \int_t^\infty g(\tau)d\tau. \tag{D.5.6}$$

Differentiating equation (D.5.1) with respect to t yields

$$\frac{d}{dt}\int_t^\infty f(\tau)d\tau = \frac{d}{dt}A(t) + \frac{d}{dt}\int_t^\infty g(\tau)d\tau$$

and using Leibniz's rule from (4.3.3) we have

$$-f(t) + \int_t^\infty \frac{d}{dt}f(\tau)d\tau = \dot{A}(t) - g(t) + \int_t^\infty \frac{d}{dt}g(\tau)d\tau.$$

Now, by employing (D.5.5), we obtain

$$-D_r(t)E(t) + \int_t^\infty \frac{d}{dt}\left(D_r(\tau)E(\tau)\right)d\tau = \dot{A}(t) - D_r(t)I(t) + \int_t^\infty \frac{d}{dt}\left(D_r(\tau)I(\tau)\right)d\tau$$

and since $E(\tau)$ and $I(\tau)$ do not depend on t, we have

$$-D_r(t)E(t) + \int_t^\infty E(\tau)\frac{d}{dt}D_r(\tau)d\tau = \dot{A}(t) - D_r(t)I(t) + \int_t^\infty I(\tau)\frac{d}{dt}D_r(\tau)d\tau.$$

Considering (D.5.3) we find

$$\frac{d}{dt}D_r(\tau) = \frac{d}{dt}\exp\left[-\int_t^\tau r(s)ds\right] = \exp\left[-\int_t^\tau r(s)ds\right]\frac{d\left[-\int_t^\tau r(s)ds\right]}{dt}$$
$$= r(t)\exp\left[-\int_t^\tau r(s)ds\right] = r(t)D_r(\tau),$$

where we used Leibniz's rule (4.3.3) again. Furthermore we notice that

$$D_r(t) = \exp\left[-\int_t^t r(s)ds\right] = \exp(0) = 1.$$

We therefore have

$$-E(t) + r(t)\int_t^\infty E(\tau)D_r(\tau)d\tau = \dot{A}(t) - I(t) + r(t)\int_t^\infty I(\tau)D_r(\tau)d\tau.$$

Using (D.5.1) for the integrals, we finally find $-E(t) + r(t)A(t) = \dot{A}(t) - I(t)$. This obviously coincides with the dynamic budget constraint (D.5.2).

Appendix E

Solution to exercises of chapter 5

E.1 Optimal consumption over an infinite horizon – ex. 1

Exercise 1 a)

The present value Hamiltonian reads

$$H^P = e^{-\rho[\tau-t]} \ln c(\tau) + \eta(\tau) [r(\tau) A(\tau) + w(\tau) - p(\tau) c(\tau)].$$

Hence, optimality conditions are

$$H_c^P = 0 \Leftrightarrow \frac{e^{-\rho[\tau-t]}}{c} = \eta p \tag{E.1.1}$$

$$\dot{\eta} = -H_A^P \Leftrightarrow \dot{\eta} = -\eta r. \tag{E.1.2}$$

Applying the logarithm to (E.1.1) yields $-\rho[\tau - t] - \ln c = \ln \eta + \ln p$. Computing the derivative with respect to time τ gives $-\rho - \frac{\dot{c}}{c} = \frac{\dot{\eta}}{\eta} + \frac{\dot{p}}{p}$ as variable are functions of time τ, i.e. $c = c(\tau), \eta = \eta(\tau)$ and $p = p(\tau)$. With (E.1.2),

$$\frac{\dot{c}}{c} = r - \frac{\dot{p}}{p} - \rho.$$

We are now going to derive the same result with the current value Hamiltonian. This one reads $H = \ln c(\tau) + \lambda(\tau)[r(\tau)A(\tau) + w(\tau) - p(\tau)c(\tau)]$. Optimality conditions are

$$H_c = 0 \Leftrightarrow \frac{1}{c} = \lambda p \tag{E.1.3}$$

$$\dot{\lambda} = \rho\lambda - H_A \Leftrightarrow \dot{\lambda} = \rho\lambda - r\lambda. \tag{E.1.4}$$

Applying the logarithm to (E.1.3) and then computing the derivative with respect to time τ gives $-\frac{\dot{c}}{c} - \frac{\dot{p}}{p} = \frac{\dot{\lambda}}{\lambda}$. Hence with (E.1.4) we have

$$-\frac{\dot{c}}{c} - \frac{\dot{p}}{p} = \rho - r \Leftrightarrow \frac{\dot{c}}{c} = r - \frac{\dot{p}}{p} - \rho.$$

Exercise 1 b)

The present value Hamiltonian reads

$$H^P = e^{-\rho[\tau-t]}u(c(\tau)) + \eta(\tau)\left[r(\tau)A(\tau) + w(\tau) - p(\tau)c(\tau)\right].$$

Hence optimality conditions are

$$H_c^P = 0 \Leftrightarrow \frac{e^{-\rho[\tau-t]}}{u'(c)} = \eta p \tag{E.1.5}$$

$$\dot{\eta} = -H_A^P \Leftrightarrow \dot{\eta} = -\eta r. \tag{E.1.6}$$

Applying the logarithm to (E.1.5) yields $-\rho[\tau-t] - \ln u'(c) = \ln \eta + \ln p$. Computing the derivative with respect to time τ gives $-\rho - \frac{u''}{u'}\dot{c} = \frac{\dot{\eta}}{\eta} + \frac{\dot{p}}{p}$ as $u(c) = u(c(\tau)), \eta = \eta(\tau)$ and $p = p(\tau)$. With (E.1.6),

$$\frac{u''}{u'}\dot{c} = r - \frac{\dot{p}}{p} - \rho.$$

Now again with the current value Hamiltonian, $H = u(c) + \lambda[rA + w - pc]$. Optimality conditions are

$$H_c = 0 \Leftrightarrow u'(c) = \lambda p \tag{E.1.7}$$

$$\dot{\lambda} = \rho\lambda - H_A \Leftrightarrow \dot{\lambda} = \rho\lambda - \lambda r. \tag{E.1.8}$$

Applying the logarithm to (E.1.7) and then computing the derivative yields $\frac{u''}{u'}\dot{c} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{p}}{p}$. With (E.1.8) we therefore obtain

$$-\frac{u''}{u'}\dot{c} = r - \frac{\dot{p}}{p} - \rho. \tag{E.1.9}$$

E.2 Optimal consumption – ex. 4

Exercise 4 a)

From the main text in (5.6.4) and exercise 1 b of ch. 5, we know the so called Keynes-Ramsey rule (E.1.9). With the given CES utility function,

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1-\sigma}, \tag{E.2.1}$$

we get $u'(\tau) = c(\tau)^{-\sigma}$ and $u''(\tau) = -\sigma c(\tau)^{-\sigma-1}$. The optimality condition then reads

$$\frac{\dot{c}}{c} = \frac{r - \frac{\dot{p}}{p} - \rho}{\sigma}.$$

Exercise 4 b)

Let us first recapitulate L'Hôpital's Rule. Let $f(x)$ and $g(x)$ be two real functions twice continuously differentiable. Suppose that the limits of both functions as x approaches x^* are 0, i.e. $\lim_{x\to x^*}[f(x)] = \lim_{x\to x^*}[g(x)] = 0$. Then L'Hôpital's rule implies

$$\lim_{x\to x^*}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to x^*}\left(\frac{f'(x)}{g'(x)}\right).$$

This rule applies to the indeterminate forms $0/0$ and ∞/∞.

Now define $f(\sigma) \equiv c^{1-\sigma} - 1$ and $g(\sigma) \equiv 1-\sigma$ and apply L'Hôpital's Rule for $\sigma^* = 1$ to the given CES utility function. We get

$$\lim_{\sigma\to 1} u(c) = \lim_{\sigma\to 1}\frac{c^{1-\sigma}-1}{1-\sigma} \overset{\text{L'Hôpital}}{=} \lim_{\sigma\to 1}\frac{c^{1-\sigma}(-\ln c)}{-1} = \ln(c),$$

where we have used

$$\frac{d}{d\sigma}c^{1-\sigma} = \frac{d}{d\sigma}e^{(1-\sigma)\ln c} = e^{(1-\sigma)\ln c}\frac{d}{d\sigma}(1-\sigma)\ln c = c^{1-\sigma}(-\ln c).$$

Exercise 4 c)

Computing the first two derivatives of the CES utility function (E.2.1) yields $u'(c(\tau)) = c(\tau)^{-\sigma} > 0$ and $u''(c(\tau)) = -\sigma c(\tau)^{-\sigma-1} < 0$. The first derivative is positive for any values of σ (and positive $c(\tau)$). The second derivative is negative for all positive σ. Hence, $\sigma > 1$ still implies decreasing marginal utility from consumption even though instantaneous utility (E.2.1) is negative for $\sigma > 1$.

Exercise 4 d)

We express the definition in (2.2.9) for t and $t+\Delta$ as $\epsilon_{t,t+\Delta} \equiv \frac{u_{c(t)}/u_{c(t+\Delta)}}{c(t+\Delta)/c(t)}\frac{d(c(t+\Delta)/c(t))}{d\left(u_{c(t)}/u_{c(t+\Delta)}\right)}$. We compute the derivatives as in (2.2.11) using the instantaneous utility function $u(c(\tau)) = \frac{c(\tau)^{1-\sigma}-1}{1-\sigma}$ from (E.2.1) as

$$\epsilon_{t,t+\Delta} \equiv \frac{\gamma[1-\sigma]c(t)^{-\sigma}/(1-\gamma)(1-\sigma)c(t+\Delta)^{-\sigma}}{c(t+\Delta)/c(t)}\frac{d(c(t+\Delta)/c(t))}{d\left(\gamma[1-\sigma]c(t)^{-\sigma}/(1-\gamma)(1-\sigma)c(t+\Delta)^{-\sigma}\right)}.$$

Following the same steps as after (2.2.11) in discrete time, we find $\epsilon_{t,t+\Delta} = 1/\sigma$.

E.3 A central planner – ex. 5

Exercise 5 a)

We start from the following setup,

$$\text{Social welfare function:} \qquad U^i(t) = \int_t^\infty e^{-\rho[\tau-t]} \frac{C^{1-\sigma}-1}{1-\sigma} d\tau, \tag{E.3.1}$$

$$\text{EU resource constraint:} \qquad \dot{K} = BK - C. \tag{E.3.2}$$

The Hamiltonian then reads $H = (C^{1-\sigma}-1)(1-\sigma)^{-1} + \lambda(BK - C)$ and the corresponding optimality conditions are

$$C^{-\sigma} = \lambda \Leftrightarrow -\sigma \ln C = \ln \lambda \Rightarrow -\sigma \frac{\dot{C}}{C} = \frac{\dot{\lambda}}{\lambda} \tag{E.3.3}$$

$$\dot{\lambda} = \rho\lambda - \lambda B \Leftrightarrow \frac{\dot{\lambda}}{\lambda} = \rho - B.$$

Combining them yields the consumption growth rate $\dot{C}/C = \sigma^{-1}(B-\rho)$.

Exercise 5 b)

Consider the linear differential equation $\dot{x}(\tau) = a(\tau)x(\tau) + b(\tau)$ and its backward solution

$$x(\tau) = x_t e^{\int_t^\tau a(s)ds} + \int_t^\tau e^{\int_s^\tau a(u)du} b(s)\, ds.$$

Given the growth rate of consumption,

$$\frac{\dot{C}}{C} = \frac{B-\rho}{\sigma} \Leftrightarrow \dot{C}(\tau) = \frac{B-\rho}{\sigma} C(\tau), \tag{E.3.4}$$

we can define $b(\tau) \equiv 0$, $a(\tau) \equiv \frac{B-\rho}{\sigma}$ and $x(\tau) \equiv C(\tau)$.The backward solution to the differential equation (E.3.4) is then obviously

$$C(\tau) = C_t e^{\frac{B-\rho}{\sigma}(\tau-t)}. \tag{E.3.5}$$

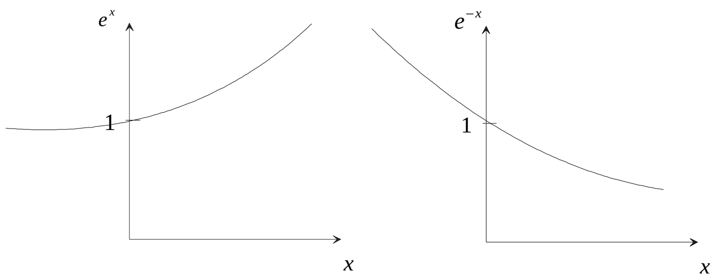
Using this for the utility function, we express the latter as

$$\begin{aligned}
U(t) &= \int_t^\infty e^{-\rho[\tau-t]}\left[\left(C_t e^{\frac{B-\rho}{\sigma}(\tau-t)}\right)^{1-\sigma}-1\right]\frac{1}{1-\sigma}d\tau \\
&= \frac{1}{1-\sigma}\int_t^\infty \left[C_t^{1-\sigma}e^{\left[\frac{B-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)} - e^{-\rho(\tau-t)}\right]d\tau \\
&= \frac{C_t^{1-\sigma}}{1-\sigma}\int_t^\infty e^{\left[\frac{B-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)}d\tau - \frac{1}{1-\sigma}\int_t^\infty \left[e^{-\rho(\tau-t)}\right]d\tau \\
&= \frac{C_t^{1-\sigma}}{1-\sigma}\int_t^\infty e^{\left[\frac{B-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)}d\tau - \frac{1}{1-\sigma}\frac{1}{-\rho}\left[e^{-\rho(\tau-t)}\right]_t^\infty \\
&= \frac{C_t^{1-\sigma}}{1-\sigma}\int_t^\infty e^{\left[\frac{B-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)}d\tau + \frac{1}{(1-\sigma)\rho}(-1) \\
&= \text{const.} + \frac{C_t^{1-\sigma}}{1-\sigma}\int_t^\infty e^{\left[\frac{B-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)}d\tau.
\end{aligned}$$

We can now conclude that the utility function is bounded if

$$\frac{B-\rho}{\sigma}(1-\sigma)-\rho<0 \Leftrightarrow B<\frac{\rho}{1-\sigma}. \tag{E.3.6}$$

As an illustration, consider these two exponential functions:



The integral from, say, 0 to infinitive of the curve e^x in the left panel is infinite as the exponential function is ever increasing and the area below the graph is therefore ever growing the further one goes "to the right", i.e. the larger x. The area of the curve e^{-x} in the right panel "can be hoped" to be finite as the distance of the graph to the horizontal axis is becoming smaller and smaller. The area below the graph is also increasing the larger the right boundary is, but it is, as one can easily show analytically, bounded from above.

A second condition, now a more economic one, one can impose on the problem is a positive growth rate, see (E.3.4). This is not necessary, however, but makes sense from an empirical perspective.

$$\sigma^{-1}(B-\rho)>0 \Leftrightarrow B>\rho. \tag{E.3.7}$$

These two constraints can be plotted into the following figure. For illustration purposes we assume that $\sigma<1$. Otherwise the result would have to be adjusted following the condition for B from (E.3.6). In the shaded area, there is growth while the utility function is still bounded.

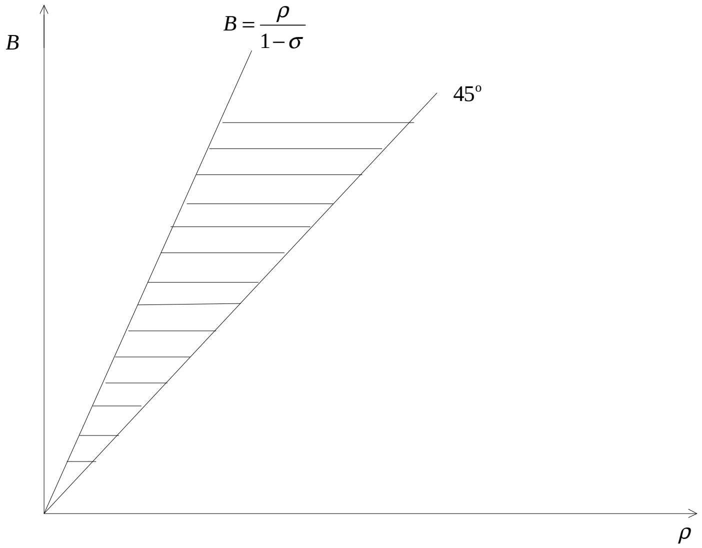


Figure E.3.1 *Boundedness condition*

Exercise 5 c)

The solution proceeds in three steps

- 1st step: Solve for the general solution of $K(\tau)$

The goods market clearing condition reads

$$\dot{K}(\tau) = BK(\tau) - C(\tau) \overset{(E.3.5)}{\Leftrightarrow} \dot{K} = BK - C_t e^{g(\tau - t)},$$

where

$$g \equiv \sigma^{-1}(B - \rho). \tag{E.3.8}$$

The backward solution is

$$K(\tau) = e^{B[\tau - t]} K_t - \int_t^\tau e^{B[\tau - s]} C_t e^{g(s-t)} ds = e^{B[\tau - t]} K_t - C_t \int_t^\tau e^{s[g-B]-gt+B\tau} ds.$$

Now, since

$$\int_t^\tau e^{s[g-B]-gt+B\tau} ds = \frac{\left[e^{s[g-B]-gt+B\tau}\right]_t^\tau}{g - B} = \frac{e^{(\tau-t)g} - e^{(\tau-t)B}}{g - B}$$

we obtain

$$K(\tau) = e^{B[\tau-t]} K_t - C_t \frac{e^{(\tau-t)g} - e^{(\tau-t)B}}{g - B}$$

and thus

$$K(\tau) = e^{(\tau-t)B}\left[K_t + \frac{C_t}{g - B}\right] - \frac{C_t e^{(\tau-t)g}}{g - B}. \tag{E.3.9}$$

- 2nd step: Use the No-Ponzi-Game condition

The No-Ponzi-Game condition for the budget constraint

$$\lim_{\tau\to\infty} e^{-\rho\tau}\lambda(\tau) K(\tau) = 0 \overset{(E.3.3)}{\Leftrightarrow} \lim_{\tau\to\infty} e^{-\rho\tau} C(\tau)^{-\sigma} K(\tau) = 0$$
$$\overset{(E.3.5)}{\Leftrightarrow} \lim_{\tau\to\infty} e^{-\rho\tau}\left[C_t e^{\sigma^{-1}(B-\rho)(\tau-t)}\right]^{-\sigma} K(\tau) = 0 \Leftrightarrow \lim_{\tau\to\infty} C_t^{-\sigma} e^{-(B-\rho)(\tau-t)-\rho\tau} K(\tau) = 0$$

- 3rd step: Insert (E.3.9) into No-Ponzi-Game condition

$$\lim_{\tau\to\infty} C_t^{-\sigma} e^{-(B-\rho)(\tau-t)-\rho\tau}\left\{e^{(\tau-t)B}\left[K_t + \frac{C_t}{g-B}\right] - \frac{C_t e^{(\tau-t)g}}{g-B}\right\} = 0$$
$$\Leftrightarrow \lim_{\tau\to\infty} C_t^{-\sigma}\left\{\left[K_t - \frac{C_t}{B-g}\right]e^{-t\rho} - \frac{C_t}{g-B} e^{(B-\rho-g)t-(B-g)\tau}\right\} = 0 \quad \text{(E.3.10)}$$

Since $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty} e^{x} = \infty$, the second term in the curly brackets only vanishes if $B - g > 0$. (Note that $(B-\rho-g)\,t$ is constant as we consider a fixed t and the limit for τ.) The difference between B and g is positive with (E.3.8) iff $B - \left(\frac{B-\rho}{\sigma}\right) > 0 \Leftrightarrow B < \frac{\rho}{1-\sigma}$. This is our condition (E.3.6).

The first term in the curly brackets only vanishes if

$$K_t = \frac{C_t}{B-g} \quad \text{(E.3.11)}$$

This implies with (E.3.9)

$$K(\tau) = \frac{C_t}{B-g} e^{(\tau-t)g} \overset{(E.3.11)}{=} K_t e^{(\tau-t)g}. \quad \text{(E.3.12)}$$

The growth rate of capital equals the consumption growth rate at every moment in time, $\dot{K}(\tau)/K(\tau) = \dot{C}(\tau)/C(\tau) = g$. This is a standard result of the Rebelo AK model.

Exercise 5 d)

You could resign from your job without making anyone less happy than before if perfect competition prevailed and in the absence of any other distortions. Since with perfect competition the decentralized economy also reaches the optimal solution. There is, therefore, no need for a central planner.

E.4 The Ramsey growth model – ex. 8

Exercise 8 a)

We start by looking at the condition for optimal consumption growth. Given the utility function $U(t) = \int_t^\infty e^{-\rho[\tau - t]} \frac{C(\tau)^{1-\sigma}-1}{1-\sigma} d\tau$ and the constraint

$$\dot{K} = Y(K, L) - C - \delta K, \tag{E.4.1}$$

the Hamiltonian reads $H = \frac{C(\tau)^{1-\sigma}-1}{1-\sigma} + \lambda(\tau)[Y(K(\tau), L) - \delta K(\tau) - C(\tau)]$. Optimality conditions are

$$H_C = C^{-\sigma} - \lambda = 0 \Leftrightarrow C^{-\sigma} = \lambda \Rightarrow -\sigma \frac{\dot{C}}{C} = \frac{\dot{\lambda}}{\lambda}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial K} = \rho\lambda - \lambda[Y_K - \delta] \Leftrightarrow \frac{\dot{\lambda}}{\lambda} = \rho - Y_K + \delta$$

Combining these two conditions gives the consumption growth rate

$$\frac{\dot{C}}{C} = \frac{Y_K - \delta - \rho}{\sigma}. \tag{E.4.2}$$

We now consider the phase diagram analysis. Equations (E.4.1) and (E.4.2) represent a two-dimensional differential equation system which, given two boundary conditions, give a unique solution for time paths $C(\tau)$ and $K(\tau)$. These two equations can be analyzed in a phase diagram.

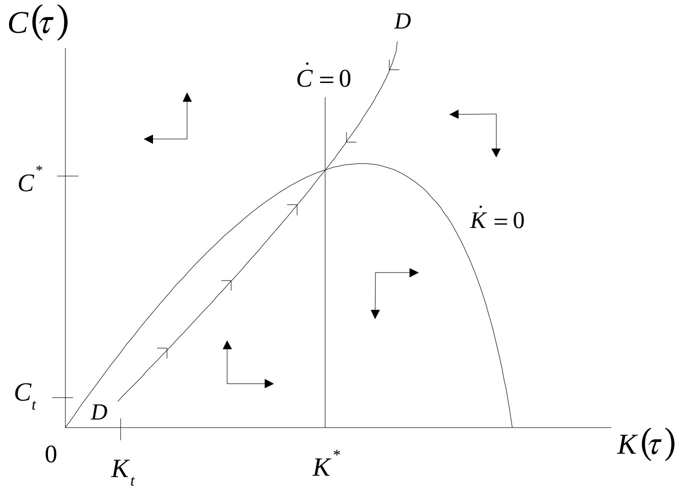
Zero-motion lines are

$$\begin{aligned} \dot{K}(\tau) &\geq 0 \Leftrightarrow C \leq Y(K, L) - \delta K, \\ \dot{C}(\tau) &\geq 0 \Leftrightarrow Y_K \geq \delta + \rho, \end{aligned}$$

for the general case. An example for a Cobb-Douglas production function $Y = K^\alpha L^{1-\alpha}$ reads

$$\dot{C}(\tau) \geq 0 \Leftrightarrow \alpha K^{\alpha-1} L^{1-\alpha} \geq \delta + \rho \Leftrightarrow K^* \leq \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}} L.$$

Plotting zero motion lines and corresponding arrow-pairs yields the following figure.



We see that the fix-point is a saddle point.

Exercise 8 b)

Concerning boundedness of the utility function, the growth rate of consumption is $\frac{\dot{C}}{C} = \frac{r-\rho}{\sigma}$, where $r \equiv Y_K - \delta$. Hence C evolves according to

$$C(\tau) = C_t e^{\frac{r-\rho}{\sigma}(\tau - t)}$$

if we are willing to assume a constant interest rate r. We can do this for the question at hand by setting r equal to its upper bound during the course of capital accumulation (i.e. its level at the initial capital stock K_0).

The utility function can then be expressed as

$$\begin{aligned} U(t) &= \int_t^\infty e^{-\rho(\tau-t)} \left[\left(C_t e^{\frac{r-\rho}{\sigma}(\tau-t)} \right)^{1-\sigma} - 1 \right] \frac{1}{1-\sigma} d\tau \\ &= \frac{1}{1-\sigma} \int_t^\infty \left[C_t^{1-\sigma} e^{\left[\frac{r-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)} - e^{-\rho(\tau-t)} \right] d\tau \\ &= \underbrace{\frac{C_t^{1-\sigma}}{1-\sigma} \int_t^\infty e^{\left[\frac{r-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)} d\tau}_{\equiv M} - \frac{1}{1-\sigma} \int_t^\infty \left[e^{-\rho(\tau-t)} \right] d\tau \end{aligned}$$

$$\Leftrightarrow \; U(t) = M - \frac{1}{1-\sigma} \frac{1}{(-\rho)} \left[e^{-\rho(\tau-t)} \right]_t^\infty = M + \frac{1}{(1-\sigma)\rho}(-1)$$

$$= -\underbrace{\frac{1}{(1-\sigma)\rho}}_{\text{const.}} + \frac{C_t^{1-\sigma}}{1-\sigma} \int_t^\infty e^{\left[\frac{r-\rho}{\sigma}(1-\sigma)-\rho\right](\tau-t)} d\tau$$

We can now conclude that the utility function is bounded if

$$\frac{r-\rho}{\sigma}(1-\sigma)-\rho<0 \Leftrightarrow r<\frac{\rho}{1-\sigma}.$$

As we assumed r to represent the highest interest rate for all capital stocks between K_0 and the steady state, this is a sufficient condition for boundedness. It is not necessary and can be relaxed.

Appendix F

Solution to exercises of chapter 6

F.1 The envelope theorem once again – ex. 1

From the main text in (6.1.4) and (6.1.5), we already know that the maximized Bellman equation is

$$\rho V(a(t)) = u(c(a(t))) + V'(a(t))[ra(t) + w - pc(a(t))].$$

Computing the derivative with respect to assets $a(t)$ yields

$$\begin{aligned}\rho V'(a(t)) = {} & u'(c(a(t)))c'(a(t)) + V''(a(t))[ra(t) + w - pc(a(t))] \\ & + V'(a(t))[r - pc'(a(t))].\end{aligned}$$

As no envelope theorem was used here, we see the derivative of the control variable with respect to the state variable.

Using the first-order condition (6.1.5), which reads $u'(c(t)) = pV'(a(t))$, yields the derivative

$$\rho V'(a(t)) = V''(a(t))[ra(t) + w - pc(a(t))] + V'(a(t))r.$$

Appendix G

Solution to exercises of chapter 7 and 8

G.1 Closed-form solution for a CRRA utility function – ex. 6

Exercise 6 a)

Let there be an utility function

$$U_t = E_t\left[\gamma c_t^{1-\sigma} + (1-\gamma)\, c_{t+1}^{1-\sigma}\right] = \gamma c_t^{1-\sigma} + (1-\gamma)\, E_t c_{t+1}^{1-\sigma}.$$

Substituting again savings from the budget constraints (8.1.6) and (8.1.7) gives an expression similar to (8.1.10), $U_t = \gamma\,[w_t - s_t]^{1-\sigma} + (1-\gamma)\, E_t\left[((1+r_{t+1})\, s_t)^{1-\sigma}\right]$, where expectations need to be formed with respect to the term in parenthesis, $((1+r_{t+1})\, s_t)^{1-\sigma}$. As only the interest rate is uncertain from the perspective of t but savings s_t are not (as it is the variable under control of the household), we can express preferences by

$$U_t = \gamma\,[w_t - s_t]^{1-\sigma} + (1-\gamma)\, s_t^{1-\sigma}\Phi$$

where $\Phi \equiv E_t\left[(1+r_{t+1})^{1-\sigma}\right]$.

The first-order condition with respect to s_t is then

$$(1-\sigma)\,\gamma\,[w_t - s_t]^{-\sigma} = (1-\sigma)\,[1-\gamma]\, s_t^{-\sigma}\Phi$$

$$\Leftrightarrow \frac{\gamma}{(w_t - s_t)^{\sigma}} = \frac{(1-\gamma)\,\Phi}{s_t^{\sigma}} \Leftrightarrow w_t - s_t = s_t\left[\frac{\gamma}{\Phi\,[1-\gamma]}\right]^{\frac{1}{\sigma}} = s_t\left\{\frac{\gamma}{(1-\gamma)\,\Phi}\right\}^{\varepsilon},$$

where the last equality defined the intertemporal elasticity of substitution $\varepsilon \equiv \frac{1}{\sigma}$.

Note the close similarity of the first-order condition here to the one in the logarithmic stochastic case in (8.1.11): The factor Φ is new here and the parameter σ which captures the degree of intertemporal elasticity of substitution. Solving for savings gives

$$s_t \left[1 - \left\{ \frac{\gamma}{(1-\gamma)\Phi} \right\}^{\varepsilon} \right] = w_t \Leftrightarrow s_t = \frac{w_t}{1 + \left\{ \frac{\gamma}{(1-\gamma)\Phi} \right\}^{\varepsilon}}. \tag{G.1.1}$$

With (8.1.7), consumption when old is then

$$c_{t+1} = [1 + r_{t+1}] s_t = (1 + r_{t+1}) \frac{w_t}{1 + \left\{ \frac{\gamma}{(1-\gamma)\Phi} \right\}^{\varepsilon}}$$

which shows that consumption in $t+1$ is uncertain indeed, as r_{t+1} is uncertain. Consumption as young is from (8.1.6)

$$c_t = w_t - s_t = \left(1 - \frac{1}{1 + \left\{ \frac{\gamma}{(1-\gamma)\Phi} \right\}^{\varepsilon}} \right) w_t = \frac{\left\{ \frac{\gamma}{(1-\gamma)\Phi} \right\}^{\varepsilon}}{1 + \left\{ \frac{\gamma}{(1-\gamma)\Phi} \right\}^{\varepsilon}} w_t.$$

Exercise 6 b)

With $\varepsilon = 1$, implying $\Phi = 1$ because of $\sigma = 1$, we recover the logarithmic solution in (8.1.12) as

$$s_t = \frac{w_t}{1 + \left\{ \frac{\gamma}{(1-\gamma)} \right\}} = \frac{w_t}{\frac{1-\gamma}{1-\gamma} + \frac{\gamma}{1-\gamma}} = [1-\gamma] w_t$$

When individuals make saving decisions in t the interest r_{t+1} they get for their savings in $t+1$ is uncertain as interest r_{t+1} and hence consumption c_{t+1} are realized at the end of period $t+1$. As people don't know what interest will be like in $t+1$ they make expectations in order to determine their savings in t.

Consumption c_t is not uncertain from the perspective t as it is realized in t. If we are in $t-1$, c_t is uncertain for the same reason why c_{t+1} in t is uncertain.

Appendix H

Solutions to exercises of chapter 9

H.1 A household maximization problem – ex. 2

We consider a standard consumption-saving problem of a household, $\max_{\{c_\tau\}_{\tau\geq t}} U_t$, where

$$U_t = E_t \Sigma_{\tau=t}^{\infty} \beta^{\tau-t} u\left(c_\tau\right) \tag{H.1.1}$$

subject to $k_{\tau+1} = (1 + r_\tau) k_\tau + \frac{w_\tau}{v_\tau} - \frac{p_\tau}{v_\tau} c_\tau$.

The household chooses his consumption path $\{c_\tau\}$ thus he maximizes his intertemporal utility U_t. We can solve this problem (H.1.1) by using dynamic programming. This solution method can be split into three parts.

- DP 1: Bellman equation and first-order condition

The Bellman equation has the following form

$$V\left(k_t\right) = \max_{c_t} \left\{u\left(c_t\right) + \beta E_t \left(V(k_{t+1})\right)\right\} \tag{H.1.2}$$

with $V(.)$ the value function of our problem (H.1.1). Therefore the first-order condition is

$$u'(c_t) = \beta E_t \left(V'(k_{t+1})\right) \frac{p_t}{v_t} \tag{H.1.3}$$

- DP 2: Evolution of the costate variable

The derivative of the Bellman equation with respect to the capital stock of the household gives (using the envelope theorem)

$$V'(k_t) = \beta E_t \left(V'(k_{t+1})\right) (1 + r_t) \tag{H.1.4}$$

- DP 3: Inserting of the first-order condition

Finally, we insert the first-order condition (H.1.3) in the equation (H.1.4) and yield the Euler equation

$$\frac{u'(c_t)}{p_t/v_t} = E_t \left[\frac{\beta u'(c_{t+1})}{(1 + r_{t+1})^{-1} p_{t+1}/v_{t+1}}\right]$$

H.2 Labour demand under adjustment cost – ex. 7

The general objective function is given by $\Pi_t = E_t \Sigma_{\tau=t}^{\infty} \frac{\pi_\tau}{[1+r]^{\tau-t}}$. Instantaneous profits amount to

$$\pi_\tau = F(L_\tau) - w_\tau L_\tau - \Phi(L_\tau - L_{\tau-1}) \tag{H.2.1}$$

where $w_\tau L_\tau$ labor cost and $\Phi(L_\tau - L_{\tau-1})$ adjustment cost in τ. The normal assumption is that both hiring and firing individuals induces costs so a typical functional form is a quadratic form like $\Phi(L_\tau - L_{\tau-1}) = \frac{\phi}{2}[L_\tau - L_{\tau-1}]^2$.

The objective function reads from inserting (H.2.1) into the objective function (9.5.6)

$$\Pi_t = E_t \Sigma_{\tau=t}^{\infty} \frac{F(L_\tau) - w_\tau L_\tau - \Phi(L_\tau - L_{\tau-1})}{(1+r)^{\tau-t}}.$$

Computing the derivative with respect to L_τ gives

$$E_\tau \left\{ \frac{F'(L_\tau) - w_\tau - \Phi'(L_\tau - L_{\tau-1})}{(1+r)^{\tau-t}} + \frac{\Phi'(L_{\tau+1} - L_\tau)}{(1+r)^{\tau+1-t}} \right\} = 0.$$

We have index τ in the expectation because when we have to decide about employment in τ we have all information available in τ. The derivative shows that choosing employment for τ affects the firm in two periods: in period τ where L_τ is chosen, where it determines output and where L_τ shows up in the adjustment cost function Φ and in period $\tau+1$, where it shows up again in the adjustment cost function when optimal employment for $\tau+1$ is chosen.

When is L_τ chosen? In τ. Hence, we form expectations in τ and all quantities with index τ are known. The first-order condition can therefore be rewritten as

$$\frac{F'(L_\tau) - w_\tau - \Phi'(L_\tau - L_{\tau-1})}{(1+r)^{\tau-t}} + \frac{E_\tau \Phi'(L_{\tau+1} - L_\tau)}{(1+r)^{\tau+1-t}} = 0.$$

There is only uncertainty about $L_{\tau+1}$ which is uncertain as the wage in $\tau+1$ is unknown and $L_{\tau+1}$ will be adjusted accordingly in $\tau+1$. Hence, expectations apply only to the first adjustment cost term and we can simplify the first-order condition further to

$$E_\tau \frac{\Phi'(L_{\tau+1} - L_\tau)}{1+r} + F'(L_\tau) - w_\tau - \Phi'(L_\tau - L_{\tau-1}) = 0.$$

Expressed for t and after rearranging, we have the equation in the text,

$$F'(L_t) = w_t + \Phi'(L_t - L_{t-1}) - E_\tau \frac{\Phi'(L_{t+1} - L_t)}{1+r}.$$

When we specify $\Phi(L_t - L_{t-1}) = \frac{\phi}{2}(L_t - L_{t-1})^2$, we obtain

$$F'(L_t) = w_t + \phi[L_t - L_{t-1}] - E_t \frac{\phi[L_{t+1} - L_t]}{1+r}.$$

Additional production must pay additional wages and additional costs today but we subtract expected adjustment costs tomorrow.

When there were no adjustment cost, i.e. $\phi = 0$, we have $F'(L_t) = w_t$. Employment is chosen such that marginal productivity equals the real wage.

H.3 Solving by substitution – ex. 8

Utility is given by $U_0 = E_0 \Sigma_{\tau=0}^{\infty} \beta^{\tau} u(c_\tau)$ and individuals maximize it with respect to

$$a_{\tau+1} = (1 + r_\tau) a_\tau + w_\tau - p_\tau c_\tau \Leftrightarrow c_\tau = \frac{(1 + r_\tau) a_\tau - a_{\tau+1} + w_\tau}{p_\tau}.$$

Replacing c_τ in the utility function gives

$$U_0 = E_0 \Sigma_{\tau=0}^{\infty} \beta^{\tau} u \left(\frac{(1 + r_\tau) a_\tau - a_{\tau+1} + w_\tau}{p_\tau} \right)$$

We now can take the derivative with respect to any $a_{\tau+1}$, because by optimizing with respect to the state variable $a_{\tau+1}$ we implicitly choose c_τ. Therefore

$$E_0 \beta^{\tau} u'(c_\tau) \frac{a_{\tau+1}}{p_\tau} = E_0 \beta^{\tau+1} u'(c_{\tau+1}) \frac{1 + r_{\tau+1}}{p_{\tau+1}}$$

H.4 Optimal training for a marathon – ex. 10

The sport event will take place in some future point in time, say in m days, i.e. in $t + m$ where t is today. So $\tau \in [t, t + m]$. There is a need for training e_τ as fitness increases with training in the following way

$$F_{\tau+1} = (1 - \delta_\tau) F_\tau + e_\tau$$

while fitness also decreases over time with depreciation rate $\delta_\tau, 0 < \delta_\tau < 1$. As the individual dislikes training, utility is decreasing in training effort e_τ, i.e. $u'(e_\tau) < 0$. But the fitter the individual is, the higher happiness of an individual $h(F_{t+m})$ in $t + m$ as it will be more successful then.

With negative marginal utility of effort we can assume either increasing or decreasing marginal utility. Both might be reasonable. In the first case, additional training causes less disutility, the argument can be that we have to warm up and then the fourth kilometer might be easier than the first kilometer. But also the second case is realistic as the first kilometer might cause less disutility than the 20th kilometer. The assumption of increasing or decreasing marginal utility will turn out to be crucial for the optimal training schedule.

Exercise 10 a)

The following objective function

$$\max_{\{e_\tau\}} U_t = E_t \left\{ \beta^m h(F_{t+m}) + \Sigma_{\tau=t}^{t+m} \beta^{\tau-t} u(e_\tau) \right\} \tag{H.4.1}$$

subject to the constraint

$$F_{\tau+1} = (1 - \delta_\tau) F_\tau + e_\tau \tag{H.4.2}$$

appears reasonable.

Exercise 10 b)

When we are interested in a training schedule in a deterministic world, we do not need to form expectations.

- DP1: Bellman equation and first-order condition for optimal effort

The optimal value (value function) of the program is given by $V(F_t) = \max U_t$. We can rewrite the objective function (H.4.1) in the following way

$$\begin{aligned} U_t &= \beta^{t+m} h(F_{t+m}) + u(e_t) + \Sigma_{\tau=t+1}^{t+m} \beta^{\tau-t} u(e_\tau) \\ &= u(e_t) + \beta \left[\beta^{m-1} h(F_{t+m}) + \Sigma_{\tau=t+1}^{t+m} \beta^{\tau-t-1} u(e_\tau) \right] = u(e_t) + \beta U_{t+1}. \end{aligned}$$

If we assume optimal behaviour and replace U_t by the optimal value $V(F_t)$, we get the Bellman equation $V(F_t) = \max_{e_t} \{u(e_t) + \beta V(F_{t+1})\}$. We calculate the first-order condition with respect to e_t, $u'(e_t) + \beta V'(F_{t+1}) \frac{\partial F_{t+1}}{\partial e_t} = 0$. From (H.4.2) we get $\partial F_{t+1}/\partial e_t = 1$ which we use for the first-order condition

$$u'(e_t) + \beta V'(F_{t+1}) = 0 \Leftrightarrow u'(e_t) = -\beta V'(F_{t+1}) \tag{H.4.3}$$

Out of the first-order condition we get the optimal effort given a specific fitness state $e_t = e_t(F_t)$

- DP2: Evolution of the costate variable

We start from the maximized BE by inserting $e_t(F_t)$

$$V(F_t) = u(e_t(F_t)) + \beta V((1-\delta) F_t + e_t(F_t))$$

The derivative of with respect to F_t is given by (using the envelope theorem)

$$V'(F_t) = \beta V'(F_{t+1}) [1 - \delta_t] \tag{H.4.4}$$

Replacing $V'(F_{t+1})$ by the first-order condition (H.4.3) gives

$$V'(F_t) = \beta \left[-\frac{u'(e_t)}{\beta} \right] [1 - \delta_t] = -u'(e_t) [1 - \delta_t]. \tag{H.4.5}$$

Shifting this by one period yields

$$V'(F_{t+1}) = -u'(e_{t+1}) [1 - \delta_{t+1}]. \tag{H.4.6}$$

- DP3: Inserting first-order conditions

Inserting (H.4.5) and (H.4.6) into (H.4.4) gives

$$-u'(e_t)[1-\delta_t] = -\beta u'(e_{t+1})[1-\delta_{t+1}][1-\delta_t] \Leftrightarrow u'(e_t) = \beta u'(e_{t+1})[1-\delta_{t+1}]$$

$$\Leftrightarrow \frac{u'(e_t)}{\beta u'(e_{t+1})} = 1-\delta_{t+1} \tag{H.4.7}$$

That is, relative marginal utilities should equal 1 minus depreciation rate.

When we look back to the assumption of increasing or decreasing marginal utility we see the effect of this assumption on the optimal training schedule by restating (H.4.7)

$$\frac{u'(e_t)}{u'(e_{t+1})} = \beta[1-\delta_{t+1}]$$

We see that the right hand side is smaller than one. It must hold, therefore, that $u'(e_t) < u'(e_{t+1})$. If we make the standard assumption that utility has negative and decreasing marginal utility (positive first and negative second derivative), we find $e_t < e_{t+1}$, i.e. we will increase our training effort day by day.

If the assumed utility function has negative and increasing marginal utility, we have $e_t > e_{t+1}$. Training effort will decrease until $t+m$.

Exercise 10 c)

Let now δ be stochastic. The objective function is given by (H.4.1)

$$\max_{\{e_\tau\}} U_t = E_t\left\{\beta^m h(F_{t+m}) + \Sigma_{\tau=t}^{t+m}\beta^{\tau-t}u(e_\tau)\right\}$$

and we can reformulate it as we did in b) above.

- DP1: Bellman Equation and first-order condition

Again, the optimal value is defined by $V(F_t) = \max_{\{e_\tau\}} U_t$ subject to $F_{\tau+1} = (1-\delta_\tau)F_\tau + e_\tau$. Similar to b) we can calculate the Bellman equation as $V(F_t) = \max_{e_t}\{u(e_t) + \beta E_t V(F_{t+1})\}$. First-order condition with respect to e_t reads

$$u'(e_t) + \beta E_t V'(F_{t+1})\frac{\partial F_{t+1}}{\partial e_t} = 0$$

With $\frac{\partial F_{t+1}}{\partial e_t} = 1$ from (H.4.2) we get

$$u'(e_t) + \beta E_t V'(F_{t+1}) = 0 \Leftrightarrow u'(e_t) = -\beta E_t V'(F_{t+1})$$

which implies $e_t = e_t(F_t)$.

- DP2: Evolution of the costate variable

By inserting $e_t = e_t(F_t)$ from DP1 we get the maximized Bellman equation

$$V(F_t) = u(e_t) + \beta E_t V(F_{t+1})$$

and $F_{t+1} = F_{t+1}(F_t, e_t(F_t))$. Calculating the derivative with respect to F_t gives

$$V'(F_t) = \beta E_t V'(F_{t+1}) \frac{\partial F_{t+1}}{\partial F_t} = \beta E_t \{V'(F_{t+1})[1-\delta_t]\}.$$

As we know δ_t in t we can put $[1-\delta_t]$ in front of the expectation and get

$$V'(F_t) = \beta [1-\delta_t] E_t V'(F_{t+1}) \tag{H.4.8}$$

Inserting first-order condition gives

$$V'(F_t) = \beta [1-\delta_t] E_t \left\{-\frac{1}{\beta} u'(e_t)\right\} = -[1-\delta_t] u'(e_t) \tag{H.4.9}$$

Shifting one period

$$V'(F_{t+1}) = -[1-\delta_{t+1}] u'(e_{t+1}) \tag{H.4.10}$$

- DP3: Inserting the first-order condition

Inserting (H.4.9) and (H.4.10) into (H.4.8) gives

$$-[1-\delta_t] u'(e_t) = \beta [1-\delta_t] E_t \{-[1-\delta_{t+1}] u'(e_{t+1})\}$$

$$\Leftrightarrow u'(e_t) = \beta E_t \{[1-\delta_{t+1}] u'(e_{t+1})\} \Leftrightarrow E_t \left\{ \frac{\beta u'(e_{t+1})}{u'(e_t)} \frac{1}{\frac{1}{[1-\delta_{t+1}]}} \right\} = 1.$$

The interpretation is standard and requires relative marginal utilities to be equal to relative marginal "cost". As in all intertemporal conditions, future values are discounted either by time-preference or by "interest" rate.

Appendix I

Solutions to exercises of chapter 10

I.1 Differentials of functions of stochastic processes II – ex. 3

Exercise 3 a)

For $F(x,y) = xy$ and $dx = f^x(x,y)\,dt + g^x(x,y)\,dq^x$, $dy = f^y(x,y)\,dt + g^y(x,y)\,dq^y$, using (10.2.9) we know that

$$\begin{aligned} dF(x,y) &= [yf^x(x,y) + xf^y(x,y)]\,dt + \{[x + g^x(x,y)]\,y - xy\}\,dq_x \\ &\quad + \{x\,[y + g^y(x,y)] - xy\}\,dq_y. \end{aligned}$$

Opening the brackets and rearranging we get

$$\begin{aligned} dF(x,y) &= [yf^x(x,y) + xf^y(x,y)]\,dt + g^x(x,y)\,y dq_x + g^y(x,y)\,x dq_y \\ &= y\underbrace{[f^x(x,y)\,dt + g^x(x,y)\,dq_x]}_{\equiv dx} + x\underbrace{[f^y(x,y)\,dt + g^y(x,y)\,dq_y]}_{\equiv dy} \\ &= ydx + xdy. \end{aligned}$$

Exercise 3 b)

For $F(x,y) = xy$ and $dx = f^x(x,y)\,dt + g^x(x,y)\,dz^x$, $dy = f^y(x,y)\,dt + g^y(x,y)\,dz^y$, using (10.2.7) we can repeat the same and see that

$$\begin{aligned} &dF(x,y) \\ &= \left\{0 + f^x(x,y)\,y + f^y(x,y)\,x + \frac{1}{2}\left[[g^x(x,y)]^2\,0 + 2\rho_{xy}g^x(x,y)\,g^y(x,y) + [g^y(x,y)]^2\,0\right]\right\} dt \\ &\quad + g^x(x,y)\,ydz^x + g^y(x,y)\,xdz^y \\ &= \left\{f^x(x,y)\,y + f^y(x,y)\,x + \rho_{xy}g^x(x,y)\,g^y(x,y)\right\} dt + g^x(x,y)\,ydz^x + g^y(x,y)\,xdz^y. \end{aligned}$$

Opening the brackets and rearranging we get

$$dF(x,y) = y\underbrace{[f^x(x,y)\,dt + g^x(x,y)\,dz^x]}_{\equiv dx} + x\underbrace{[f^y(x,y)\,dt + g^y(x,y)\,dz^y]}_{\equiv dy}$$
$$+\rho_{xy}g^x(x,y)\,g^y(x,y)\,dt = ydx + xdy + \rho_{xy}g^x(x,y)\,g^y(x,y)\,dt.$$

Hence, $dF(x,y) = xdy + ydx + \rho_{xy}g^x(x,y)\,g^y(x,y)\,dt$. This tells us that $dF(x,y) = xdy + ydx$ will hold only if $\rho_{xy} = 0$, i.e. when $x(t)$ and $y(t)$ are independent.

I.2 Correlated jump processes – ex. 4

Exercise 4 a)

We are looking for an economic interpretation of labour productivity in sector X following $dA(t) = \alpha dt + \beta dq_x(t)$ and labour productivity in sector Y following $dB(t) = \gamma dt + \delta dq_y(t)$. Both q_x and q_y are linear combinations of the q_i. As q_1 appears in both processes, the sector specific shocks dA and dB are what could be called “correlated Poisson processes”. Note, however, that strictly speaking q_x and q_y are not Poisson processes any more. Shocks can be classified into aggregate shocks to the economy as a whole, q_1, and idiosyncratic sector specific shocks q_2 and q_3.

Exercise 4 b)

We now consider the evolution of GDP in terms of dq_x and dq_y. GDP is given by $G(A(t), B(t)) \equiv p_x A(t)\,L_x + p_y B(t)\,L_y$. Now compute the differential of the GDP $G(t)$. After having replaced q_x and q_y in dA and dB

$$dA(t) = \alpha dt + \beta dq_x(t) = \alpha dt + \beta d[q_1(t) + q_2(t)],$$

$$dB(t) = \gamma dt + \delta dq_y(t) = \gamma dt + \delta d[q_1(t) + q_3(t)],$$

we apply lemma 10.2.5 to get

$$\begin{aligned} dG(A(t), B(t)) &= (\alpha G_A + \gamma G_B)\,dt + \underbrace{[G(A(t)+\beta, B(t)+\delta) - G(A(t), B(t))]\,dq_1}_{\text{common shock}} \\ &\quad + \underbrace{[G(A(t)+\beta, B(t)) - G(A(t), B(t))]\,dq_2}_{\text{sector specific shock in } X} \\ &\quad + \underbrace{[G(A(t), B(t)+\delta) - G(A(t), B(t))]\,dq_3}_{\text{sector specific shock in } Y}. \end{aligned}$$

Now define as a shortcut

$$G(A,B) \equiv A\underbrace{x}_{\equiv p_x L} + B\underbrace{y}_{\equiv p_y L}.$$

With this definition we can rearrange $dG(A,B)$ to get

$$\begin{aligned} dG(A,B) &= \{\alpha x+\gamma y\}dt+\{(A+\beta)x+(B+\delta)y-(Ax+By)\}dq_1 \\ &\quad +\{(A+\beta)x+By-(Ax+By)\}dq_2 \\ &\quad +\{Ax+(B+\delta)y-(Ax+By)\}dq_3 \\ &= \{\alpha x+\gamma y\}dt+\{\beta x+\delta y\}dq_1+\beta x dq_2+\delta y dq_3 \\ &= \{\alpha x+\gamma y\}dt+\beta x(dq_1+dq_2)+\delta y(dq_1+dq_3) \\ &= \{\alpha x+\gamma y\}dt+\beta x dq_x+\delta y dq_y. \end{aligned}$$

Replacing x and y by $p_x L_x$ and $p_y L_y$ respectively yields

$$dG(A,B)=\{\alpha p_x L_x+\gamma p_y L_y\}dt+\beta p_x L_x dq_x+\delta p_y L_y dq_y.$$

I.3 Deriving a budget constraint – ex. 5

Exercise 5 a)

The change in the number of shares of stock i held by the household is the change of the fraction of savings of the household reserved for the acquisition of the stock i divided by the price of one share of this stock.

Exercise 5 b)

Prices of assets follow

$$\begin{aligned} dv_1 &= \alpha_1 v_1 dt+\beta_1 v_1 dq_1, \\ dv_2 &= \alpha_2 v_2 dt+\beta_2 v_2 dq_2, \end{aligned}$$

where q_1 and q_2 are two Poisson processes. We define savings of the household as $s \equiv \pi_1 n_1+\pi_2 n_2+w-pc$. The number of shares of asset i is determined by

$$dn_1=\chi\frac{s}{v_1}dt, \quad dn_2=(1-\chi)\frac{s}{v_2}dt.$$

Wealth of the household is defined by $a=n_1 v_1+n_2 v_2$.

The budget constraint of the household follows from the differential of a. Using the appropriate CVF yields

$$\begin{aligned} da &= \left\{n_1\alpha_1 v_1+n_2\alpha_2 v_2+v_1\chi\frac{s}{v_1}+v_2(1-\chi)\frac{s}{v_2}\right\}dt \\ &\quad +\{n_1[v_1+\beta_1 v_1]+n_2 v_2-(n_1 v_1+n_2 v_2)\}dq_1 \\ &\quad +\{n_1 v_1+n_2[v_2+\beta_2 v_2]-(n_1 v_1+n_2 v_2)\}dq_2 \\ &= \{\alpha_1 n_1 v_1+\alpha_2 n_2 v_2+s\}dt+\Sigma_{i=1}^2\beta_i n_i v_i dq_i. \end{aligned}$$

Inserting savings and collecting terms gives

$$da = \left\{ n_1 v_1 \left[\alpha_1 + \frac{\pi_1}{v_1} \right] + n_2 v_2 \left[\alpha_2 + \frac{\pi_2}{v_2} \right] + w - pc \right\} dt + \Sigma_{i=1}^2 \beta_i n_i v_i dq_i.$$

Let $\theta \equiv \frac{n_1 v_1}{a}$ and define $r_i \equiv \alpha_i + \frac{\pi_1}{v_i}$. Then

$$\begin{aligned} da &= \{a\theta r_1 + a(1-\theta) r_2 + w - pc\} dt + \theta a \beta_1 dq_1 + (1-\theta) a \beta_2 dq_2 \\ &= \{[\theta (r_1 - r_2) + r_2] a + w - pc\} dt + a [\theta \beta_1 dq_1 + (1-\theta) \beta_2 dq_2]. \end{aligned}$$

I.4 Solving stochastic differential equations – ex. 12

Exercise 12 a)

We start from

$$\begin{aligned} dx(t) &= [a(t) - x(t)] dt + B_1(t)x(t)dW_1(t) + b_2(t)dW_2(t) \\ &= [-x(t) + a(t)] dt + \{B_1(t)x(t) + 0\} dW_1(t) + \{0 + b_2(t)\} dW_2(t). \end{aligned}$$

Applying (10.4.6),

$$\begin{aligned} \Sigma_{BB} &= \rho_{11} B_1(t) B_1(t) = B_1^2(t), \\ \Sigma_{Bb} &= \rho_{12} B_1(t) b_2(t), \end{aligned}$$

gives

$$\phi(t) = \exp\left\{ \int_0^t \left(-1 - \frac{1}{2} B_1^2(s) \right) ds + \int_0^t B_1(s) dW_1(s) \right\} \Rightarrow \qquad \text{(I.4.1a)}$$

$$x(t) = \phi(t) \left[x_0 + \int_0^t \frac{a(s) - \rho_{12} B_1(s) b_2(s)}{\phi(s)} ds + \int_0^t \frac{b_2(s) dW_2(s)}{\phi(s)} \right]. \qquad \text{(I.4.1b)}$$

Exercise 12 b)

To check that the above expression for $x(t)$ is a solution define

$$\begin{aligned} y(t) &= \int_0^t \left(-1 - \frac{1}{2} B_1^2(s) \right) ds + \int_0^t B_1(s) dW_1(s) \\ z(t) &= x_0 + \int_0^t \frac{a(s) - \rho_{12} B_1(s) b_2(s)}{\phi(s)} ds + \int_0^t \frac{b_2(s) dW_2(s)}{\phi(s)} \end{aligned}$$

and get

$$x(y(t), z(t)) = e^{y(t)} z(t).$$

Applying Ito's Lemma to $x(y(t), z(t))$ we obtain

$$\begin{aligned} dx &= \underbrace{e^{y(t)}z(t)}_{=x(y(t),z(t))} dy(t) + e^{y(t)}dz(t) + \frac{1}{2}\left[\underbrace{e^{y(t)}z(t)}_{=x(y(t),z(t))} (dy(t))^2 + 2e^{y(t)}dy(t)dz(t)\right] \\ &= x(y(t), z(t))\left[dy(t) + \frac{1}{2}(dy(t))^2\right] + e^{y(t)}dz(t) + e^{y(t)}dy(t)dz(t) \end{aligned}$$

Consider each of the differentials and their products separately:

$$\begin{aligned} dy(t) &= \left(-1 - \frac{1}{2}B_1^2(t)\right)dt + B_1(t)dW_1(t) \\ (dy(t))^2 &= B_1^2(t)\,(dW_1(t))^2 = B_1^2(t)dt \\ dz(t) &= \frac{a(t) - \rho_{12}B_1(t)b_2(t)}{\phi(t)}dt + \frac{b_2(t)dW_2(t)}{\phi(t)} \\ dy(t)dz(t) &= \frac{B_1(t)b_2(t)dW_1(t)dW_2(t)}{\phi(t)} = \frac{\rho_{12}B_1(t)b_2(t)}{\phi(t)}dt. \end{aligned}$$

Substituting these back into the expression for dx we get

$$\begin{aligned} dx &= x(y(t), z(t))\left[dy(t) + \frac{1}{2}(dy(t))^2\right] + \underbrace{e^{y(t)}}_{=\phi(t)}dz(t) + \underbrace{e^{y(t)}}_{=\phi(t)}dy(t)dz(t) \\ &= x(y(t), z(t))\left[\left(-1 - \frac{1}{2}B_1^2(t)\right)dt + B_1(t)dW_1(t) + \frac{1}{2}B_1^2(t)dt\right] \\ &\quad +\phi(t)\left[\frac{a(t) - \rho_{12}B_1(t)b_2(t)}{\phi(t)}dt + \frac{b_2(t)dW_2(t)}{\phi(t)}\right] + \phi(t)\frac{\rho_{12}B_1(t)b_2(t)}{\phi(t)}dt \\ &= x(y(t), z(t))\left[-dt + B_1(t)dW_1(t)\right] \\ &\quad + \left[a(t) - \rho_{12}B_1(t)b_2(t)\right]dt + b_2(t)dW_2(t) + \rho_{12}B_1(t)b_2(t) \\ &= B_1(t)dW_1(t)x(y(t), z(t)) - x(y(t), z(t))dt + a(t)dt + b_2(t)dW_2(t) \\ &= [a(t) - x(t)]dt + B_1(t)x(t)dW_1(t) + b_2(t)dW_2(t) = dx(t). \end{aligned}$$

The very last equality shows us that Ito's Lemma is satisfied, so (I.4.1a)-(I.4.1b) is a solution.

I.5 Dynamic and intertemporal budget constraints - Brownian Motion – ex. 13

Exercise 13 a)

The dynamic budget constraint we study is

$$da(\tau) = [r(\tau)a(\tau) + w(\tau) - p(\tau)c(\tau)]\,dt + \sigma a(\tau)dz(\tau).$$

To solve for $a(\tau)$, apply the result given in (10.4.6). With $A(\tau) = r(\tau)$, $\Sigma_{BB}(\tau) = \sigma^2$ and $\Sigma_{Bb}(\tau) = 0$, we get $\phi(\tau) = e^{\int_0^\tau \left(r(u) - \frac{1}{2}\sigma^2\right)du + \int_0^\tau \sigma dz(u)}$. The solution to the above differential equation becomes $a(\tau) = \phi(\tau)\left[a(0) + \int_0^\tau \phi^{-1}(s)\left[w(s) - p(s)c(s)\right]ds\right]$ which implies

$$a(0) = a(\tau)\phi^{-1}(\tau) - \int_0^\tau \phi^{-1}(s)\left[w(s) - p(s)c(s)\right]ds.$$

Replacing 0 by t we obtain

$$a(t) = a(\tau)e^{-\int_t^\tau \left(r(u) - \frac{1}{2}\sigma^2\right)du - \int_t^\tau \sigma dz(u)} - \int_t^\tau e^{-\int_t^s \left(r(u) - \frac{1}{2}\sigma^2\right)du - \int_t^s \sigma dz(u)}\left[w(s) - p(s)c(s)\right]ds.$$

Applying to the latter result the "no-Ponzi-game" condition

$$\lim_{\tau \to \infty}\left[a(\tau)e^{-\int_t^\tau \left(r(u) - \frac{1}{2}\sigma^2\right)du - \int_t^\tau \sigma dz(u)}\right] = 0$$

we get

$$a(t) = \int_t^\infty e^{-\int_t^s \left(r(u) - \frac{1}{2}\sigma^2\right)du - \int_t^s \sigma dz(u)}\left[p(s)c(s) - w(s)\right]ds.$$

Rearranging this equation gives us the final result

$$\int_t^\infty e^{-\varphi(s)}p(s)c(s)ds = a(t) + \int_t^\infty e^{-\varphi(s)}w(s)ds,$$

where $\varphi(s) = \int_t^s \left(r(u) - \frac{1}{2}\sigma^2\right)du + \int_t^s \sigma dz(u)$.

Exercise 13 b)

We start from the BC (10.6.2) and the definition of $\varphi(\tau) = \int_t^\tau r(s) - \frac{1}{2}\sigma^2 ds + \int_t^\tau \sigma dz(\tau)$. We write the BC as

$$\int_t^\infty e^{-\int_t^\tau r(s)ds}e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)}p(\tau)c(\tau)\,d\tau = a_t + \int_t^\infty e^{-\int_t^\tau r(s)ds}e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)}w(\tau)\,d\tau.$$

Applying expectations on both sides, we get

$$E_t\int_t^\infty e^{-\int_t^\tau r(s)ds}e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)}p(\tau)c(\tau)\,d\tau = a_t + E_t\int_t^\infty e^{-\int_t^\tau r(s)ds}e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)}w(\tau)\,d\tau.$$

Now focus on the left-hand side. Pulling the expectation inside and making an independence assumption (which is actually not needed - without making it, we have the covariance term which later would disappear again when pulling the expectations out again) gives

$$\begin{aligned} & E_t \int_t^\infty e^{-\int_t^\tau r(s)ds} e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)} p(\tau) c(\tau) d\tau \\ = & \int_t^\infty E_t e^{-\int_t^\tau r(s)ds} E_t e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)} E_t p(\tau) c(\tau) d\tau \\ = & \int_t^\infty E_t e^{-\int_t^\tau r(s)ds} E_t e^{\frac{1}{2}\sigma^2[\tau-t] - \sigma[z(\tau)-z(t)]} E_t p(\tau) c(\tau) d\tau. \end{aligned}$$

Now let us focus on the expected term in the middle and let's pull out the deterministic part. This gives

$$E_t e^{\frac{1}{2}\sigma^2[\tau-t]-\sigma[z(\tau)-z(t)]} = e^{\frac{1}{2}\sigma^2[\tau-t]} E_t e^{-\sigma[z(\tau)-z(t)]}$$

Remember that $z(\tau) - z(t)$ is Brownian motion with mean zero and variance $\tau - t$. Then $-\sigma[z(\tau) - z(t)]$ is $N(0, \sigma^2[\tau - t])$ and $e^{-\sigma[z(\tau)-z(t)]}$ is lognormally distributed with mean and variance given by (7.3.4). Hence, $E_t e^{-\sigma[z(\tau)-z(t)]} = e^{\frac{1}{2}\sigma^2[\tau-t]}$ and we get

$$E_t e^{\frac{1}{2}\sigma^2[\tau-t]-\sigma[z(\tau)-z(t)]} = e^{\frac{1}{2}\sigma^2[\tau-t]} e^{\frac{1}{2}\sigma^2[\tau-t]} = e^{\sigma^2[\tau-t]}.$$

Using $E_t e^{-\int_t^\tau r(s)ds} e^{\sigma^2[\tau-t]} = E_t e^{-\int_t^\tau r(s)-\sigma^2 ds}$, the left-hand side therefore reads

$$\begin{aligned} E_t \int_t^\infty e^{-\int_t^\tau r(s)ds} e^{\int_t^\tau \frac{1}{2}\sigma^2 ds - \int_t^\tau \sigma dz(\tau)} p(\tau) c(\tau) d\tau &= \int_t^\infty E_t e^{-\int_t^\tau r(s)-\sigma^2 ds} E_t p(\tau) c(\tau) d\tau \\ &= E_t \int_t^\infty e^{-\int_t^\tau r(s)-\sigma^2 ds} p(\tau) c(\tau) d\tau. \end{aligned}$$

When we do the same steps on the RHS, we end up with

$$E_t \int_t^\infty e^{-\int_t^\tau r(s)-\sigma^2 ds} p(\tau) c(\tau) d\tau = a_t + E_t \int_t^\infty e^{-\int_t^\tau r(s)-\sigma^2 ds} w(\tau) d\tau$$

which is the BC we were looking for.

Appendix J

Solution to exercises of chapter 11

J.1 Optimal saving under Poisson uncertainty with two state variables – ex. 1

Exercise 1 a)

We view this as a problem with two state variables. Hence, the BE is a generalization of (11.1.4),

$$\rho V(a(t), p(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t), p(t)) \right\} \tag{J.1.1}$$

noting that there are now two state variables: the budget $a(t)$ and the price $p(t)$ of the consumption good as the price is no more constant evolving according to

$$dp(t) = p(t)[gdt + \sigma dq(t)] \tag{J.1.2}$$

- DP 1: First-order condition

The value of the optimal program changes because of changes in wealth due to a Poisson process dq or because of price changes dp. So the differential $dV(a(t), p(t))$ is given by

$$\begin{aligned} dV(a, p) &= \underbrace{V_a(a, p)[ra + w - pc]dt + V_p(a, p)pgdt}_{\text{deterministic change}} \\ &\quad + \underbrace{[V((1+\beta)a, (1+\sigma)p) - V(a, p)]dq}_{\text{stochastic change (jump)}} \end{aligned} \tag{J.1.3}$$

This inserted in the Bellman equation yields

$$\begin{aligned} \rho V(a(t), p(t)) &= \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t), p(t)) \right\} \\ &= \max_{c(t)} \{ u(c(t)) + V_a(a, p)[ra + w - pc] + V_p(a, p)gp \\ &\quad + \lambda [V((1+\beta)a, (1+\sigma)p) - V(a, p)] \} \end{aligned} \tag{J.1.4}$$

The first-order condition is

$$u_c(c) = V_a(a,p)\,p \tag{J.1.5}$$

So current utility from an additional unit of consumption $u_c(c)$ must equal (future) utility from an additional unit of wealth $V_a(a,p)$ multiplied by the price p of the consumption good.

- DP 2: Evolution of costate variable

Starting from the maximized Bellman equation

$$\begin{aligned}\rho V(a,p) &= u(c(a)) + V_a(a,p)\,[ra + w - pc(a)] + V_p(a,p)\,gp \\ &\quad + \lambda\,[V((1+\beta)\,a,(1+\sigma)\,p) - V(a,p)]\,,\end{aligned}$$

the derivative with respect to a is

$$\begin{aligned}\rho V_a(a,p) &= rV_a(a,p) + V_{aa}(a,p)\,[ra + w - pc(a)] + V_{pa}(a,p)\,gp \\ &\quad + \lambda\,[V_{\tilde{a}}(\tilde{a},\tilde{p})\,[1+\beta] - V_a(a,p)]\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\ & V_{aa}(a,p)\,[ra + w - pc(a)] = (\rho - r)\,V_a(a,p) \\ & -V_{pa}(a,p)\,gp - \lambda\,[V_{\tilde{a}}(\tilde{a},\tilde{p})\,[1+\beta] - V_a(a,p)]\end{aligned} \tag{J.1.6}$$

where $\tilde{a} \equiv (1+\beta)\,a$ and $\tilde{p} \equiv (1+\sigma)\,p$.

The second step is the calculation of the differential of the partial derivative of the value function

$$\begin{aligned}dV_a(a,p) &= \{V_{aa}(a,p)\,[ra + w - pc] + V_{ap}(a,p)\,gp\}\,dt \\ &\quad + [V_{\tilde{a}}((1+\beta)\,a,(1+\sigma)\,p) - V_a(a,p)]\,dq\end{aligned}$$

$$\begin{aligned}\overset{(J.1.6)}{\Leftrightarrow}\ dV_a(a,p) &= \{[\rho - r]\,V_a(a,p) - V_{pa}(a,p)\,gp \\ &\quad - \lambda\,[V_{\tilde{a}}(\tilde{a},\tilde{p})\,(1+\beta) - V_a(a,p)] + V_{ap}(a,p)\,gp\}\,dt \\ &\quad + [V_{\tilde{a}}(\tilde{a},\tilde{p}) - V_a(a,p)]\,dq\end{aligned} \tag{J.1.7}$$

where $V_{pa}(a,p)\,gp$ and $V_{ap}(a,p)\,gp$ eliminate as $V_{pa}(a,p)) = V_{ap}(a,p)$. So

$$\begin{aligned}\overset{(J.1.6)}{\Leftrightarrow}\ dV_a(a,p) &= \{[\rho - r]\,V_a(a,p) - \lambda\,[V_{\tilde{a}}(\tilde{a},\tilde{p})\,(1+\beta) - V_a(a,p)]\}\,dt \\ &\quad + [V_{\tilde{a}}(\tilde{a},\tilde{p}) - V_a(a,p)]\,dq\end{aligned} \tag{J.1.8}$$

- DP 3: Inserting first-order conditions

Now we can replace the marginal value by marginal utility from the first-order condition (J.1.5) using the appropriate CVF formula: Let $F(x,y)$ be a function of the two Poisson processes $dx = a(x,y)\,dt + b(x,y)\,dq$ and $dy = c(x,y)\,dt + g(x,y)\,dq$. So generally the differential of $F(x,y)$ is given by

$$\begin{aligned} dF(x,y) &= \{F_x a(x,y) + F_y c(x,y)\}\,dt \\ &\quad + \{F(x+b(x,y), y+g(x,y)) - F(x,y)\}\,dq \end{aligned}$$

Applying this to our two Poisson processes dV_a and dp given by (J.1.8) and (J.1.2) yields:

$$\begin{aligned} du_c(c) &= \{p[[\rho - r]V_a(a,p) - \lambda[V_{\widetilde{a}}(\widetilde{a},\widetilde{p})(1+\beta) - V_a(a,p)]] + V_a(a,p)pg\}\,dt \\ &\quad + [V_{\widetilde{a}}(\widetilde{a},\widetilde{p}) - V_a(a,p)]\,dq \\ &= \{p[[\rho + g - r]V_a(a,p) - \lambda[V_{\widetilde{a}}(\widetilde{a},\widetilde{p})(1+\beta) - V_a(a,p)]]\}\,dt \\ &\quad + [V_{\widetilde{a}}(\widetilde{a},\widetilde{p})\widetilde{p} - V_a(a,p)p]\,dq \end{aligned}$$

Exercise 1 b)

This follows the same steps as the derivation of the Keynes-Ramsey rule in (11.1.10).

Exercise 1 c)

When either β or σ or both are zero, this affects the jumps of wealth and the price, $\widetilde{a} \equiv (1+\beta)a$ and $\widetilde{p} \equiv (1+\sigma)p$. The derivation is therefore identical to a) only that β or σ or both are zero.

J.2 Optimal saving under Brownian motion – ex. 2

Here, dp/p is assumed to be deterministic, i.e. there are only deterministic price changes $dp(t) = p(t)\,gdt$. The general Keynes-Ramsey rule for marginal utility is given from (11.3.4),

$$\begin{aligned} du'(c) &= \left\{(\rho - r)u'(c) - u''(c)c_a(t)\beta^2 a(t)\right\}dt \\ &\quad + u''(c)c_a(t)\beta a(t)\,dz(t) + u'(c)\,dp(t)/p(t) \end{aligned} \tag{J.2.1}$$

$$\Leftrightarrow\ du'(c) = \left\{(\rho - r)u'(c) - u''(c)c_a(t)\beta^2 a(t)\right\}dt$$
$$+u''(c)c_a(t)\beta a(t)\,dz(t) + u'(c)\frac{p(t)g}{p(t)}dt$$

$$\Leftrightarrow\ du'(c) = \underbrace{\left\{(\rho + g - r)u'(c) - u''(c)c_a(t)\beta^2 a(t)\right\}}_{x}dt$$
$$+\underbrace{u''(c)c_a(t)\beta a(t)}_{y}dz(t)$$

- Derivation of dc

Let $f(.)$ be the inverse function for u_c, i.e. $f(u'(c)) = c$. For the first derivative of the inverse function we apply the chain rule and get

$$\frac{df(u'(c))}{dc} = \underbrace{\frac{df(u'(c))}{du'(c)}}_{w}\underbrace{\frac{du'(c)}{dc}}_{v} = 1$$

Hence,

$$\frac{df(u'(c))}{du'(c)} = \frac{1}{\frac{du'(c)}{dc}} \tag{J.2.2}$$

$$= \frac{1}{u_{cc}} \tag{J.2.3}$$

Now calculate the second derivative of $f(u'(c)) = c$ with respect to c applying the chain rule once again as well as the product rule

$$\frac{d^2f(u'(c))}{dc^2} = w'v + w'u$$

$$= \underbrace{\frac{d^2f(u'(c))}{d(u'(c))^2}\frac{du'(c)}{dc}}_{w'}\underbrace{\frac{du'(c)}{dc}}_{v} + \underbrace{\frac{d^2u'(c)}{dc^2}}_{v'}\underbrace{\frac{df(u'(c))}{du'(c)}}_{w} = 0$$

$$\Leftrightarrow \frac{d^2f(u'(c))}{d(u'(c))^2} = -\frac{\frac{d^2u'(c)}{dc^2}\frac{df(u'(c))}{du'(c)}}{\left[\frac{du'(c)}{dc}\right]^2} \Leftrightarrow \frac{d^2f(u'(c))}{d(u'(c))^2} \overset{(J.2.2)}{=} -\frac{u'''(c)\frac{1}{\frac{du'(c)}{dc}}}{\left[\frac{du'(c)}{dc}\right]^2}$$

$$\Leftrightarrow \frac{d^2f(u'(c))}{d(u'(c))^2} = -\frac{u'''(c)}{[u''(c)]^3} \tag{J.2.4}$$

Now applying Ito's Lemma (see chapter 9.2.2, p. 172) to $f(u_c)$ yields

$$df(u'(c)) = f_t dt + f_{u_c}x(.)dt + f_{u_c}y(.)dz + \frac{1}{2}f_{u_cu_c}y^2(.)dt$$

$$= \left\{f_t + f_{u_c}x(.) + \frac{1}{2}f_{u_cu_c}y(.)^2\right\}dt + f_{u_c}y(.)dz$$

$$\overset{(J.2.1)}{=} \left\{\frac{1}{u''(c)}\left[(\rho + g - r)u'(c) - u''(c)c_a(t)\beta^2 a(t)\right]\right.$$

$$\left. -\frac{1}{2}\frac{u'''(c)}{[u''(c)]^3}\left[u''(c)c_a(t)\beta a(t)\right]^2\right\}dt$$

$$+\frac{1}{u''(c)}u''(c)c_a(t)\beta a(t)dz$$

$$\Leftrightarrow \quad dc = -\frac{u'(c)}{u''(c)}\left\{r+g-\rho+\frac{u''(c)}{u'(c)}c_a(t)\beta^2 a(t)+\frac{1}{2}\frac{u'''(c)}{u'(c)}\left[c_a(t)\beta a(t)\right]^2\right\}dt$$
$$+c_a(t)\beta a(t)\,dz$$

$$\Leftrightarrow \quad -\frac{u''(c)}{u'(c)}dc = \left\{r+g-\rho+\frac{u''(c)}{u'(c)}c_a(t)\beta^2 a(t)+\frac{1}{2}\frac{u'''(c)}{u'(c)}\left[c_a(t)\beta a(t)\right]^2\right\}dt$$
$$-\frac{u''(c)}{u'(c)}c_a(t)\beta a(t)\,dz \qquad \text{(J.2.5)}$$

- Keynes-Ramsey rule for utility function $u(c(t)) = \ln c(t)$

Given the utility function $u(c(t)) = \ln c(t)$, we know that $u'(c) = \frac{1}{c}$, $u''(c) = -\frac{1}{c^2}$ and $u'''(c) = 2\frac{1}{c^3}$. Using this in (J.2.5) yields

$$\frac{c}{c^2}dc = \left\{r+g-\rho-\frac{c}{c^2}c_a(t)\beta^2 a(t)+\frac{1}{2}\frac{2\frac{1}{c^3}}{\frac{1}{c}}\left[c_a(t)\beta a(t)\right]^2\right\}dt$$
$$+\frac{c}{c^2}c_a(t)\beta a(t)\,dz$$

$$\frac{dc}{c} = \left\{r+g-\rho-\frac{1}{c}c_a(t)\beta^2 a(t)+\frac{1}{c^2}\left[c_a(t)\beta a(t)\right]^2\right\}dt$$
$$-\frac{1}{c}c_a(t)\beta a(t)\,dz$$

J.3 Adjustment cost – ex. 3

The maximization problem of the firm has the following structure,

$$\max_{(i(\tau))_{\tau\geq t}} \Pi_t = E_t \int_t^\infty e^{-r(\tau-t)}\left[pk(\tau)^\alpha - ci(\tau)^2\right]d\tau \qquad \text{(J.3.1)}$$
$$s.t. \quad dk(\tau) = (i(\tau)-\delta k(\tau))\,d\tau \qquad \text{(J.3.2)}$$
$$dc(\tau) = \alpha_c c(\tau)d\tau + \sigma_c c(\tau)dz_c(\tau) \qquad \text{(J.3.3)}$$
$$dp(\tau) = \alpha_p p(\tau)d\tau + \sigma_p p(\tau)dz_p(\tau) \qquad \text{(J.3.4)}$$
$$k(t) = k_t,\ c(t) = c_t,\ p(t) = p_t. \qquad \text{(J.3.5)}$$

Exercise 3 a)

Here we assume $\alpha_p = \sigma_p = 0$. Therefore the price is $p(\tau) = p_t$ in each point in time τ. Now we can solve the maximization problem (J.3.1) - (J.3.5) by using dynamic programming.

- DP 1: Bellman equation and first-order condition

In the first step we have to determine the Bellman equation of the maximization problem. Therefore we use Ito's Lemma (10.2.2), thus we yield the Bellman equation

$$\begin{aligned} rV(k,c) &= \max_{i(t)} \left\{ pk^\alpha - ci^2 + \frac{E_t V(k,c)}{dt} \right\} \\ &= \max_{i(t)} \left\{ pk^\alpha - ci^2 + V_k [i - \delta k] + V_c \alpha_c c + \frac{1}{2} V_{cc} \sigma_c^2 c^2 \right\}. \end{aligned} \tag{J.3.6}$$

The first-order condition according to the Bellman equation (J.3.6) has the form

$$V_k = 2ci \tag{J.3.7}$$

implying

$$V_{kc} = 2i. \tag{J.3.8}$$

- DP 2: Evolution of the costate variable

The derivative of the maximized Bellman equation (J.3.6) with respect to the state variable k gives (using the Envelope theorem)

$$\begin{aligned} & rV_k = \alpha \frac{p}{k^{1-\alpha}} + V_{kk}(i - \delta k) - \delta V_k + V_{kc} \alpha_c c + \frac{1}{2} V_{kcc} \sigma_c^2 c^2 \\ & \Leftrightarrow V_{kk}(i - \delta k) + V_{kc} \alpha_c c + \frac{1}{2} V_{kcc} \sigma_c^2 c^2 = (r + \delta) V_k - \alpha \frac{p}{k^{1-\alpha}} \end{aligned} \tag{J.3.9}$$

Computing the differential of the shadow price of capital V_k gives, using Ito's Lemma (10.2.2)

$$\begin{aligned} dV_k &= V_{kk}(i - \delta k) \, dt + V_{kc} (\alpha_c c dt + \sigma_c c dz_c) + \frac{1}{2} V_{kcc} \sigma_c^2 c^2 dt \\ &= V_{kk} \left[(i - \delta k) + V_{kc} \alpha_c c + \frac{1}{2} V_{kcc} \sigma_c^2 c^2 \right] dt + \sigma_c c dz_c \end{aligned} \tag{J.3.10}$$

If we insert (J.3.9) into (J.3.10) we get

$$dV_k = \left[(r + \delta) V_k - \alpha_c p k^{\alpha - 1} \right] dt + V_{kc} \sigma_c c dz_c \tag{J.3.11}$$

- DP 3: Inserting the first-order condition

Now we replace V_k by (J.3.7) and V_{kc} by (J.3.8) in equation (J.3.11). Therefore we get

$$\begin{aligned} 2dci &= \left[(r + \delta - \alpha_c) 2ci - \alpha p k^{\alpha - 1} \right] dt + 2\sigma ci dz_c \\ \Leftrightarrow_{b \equiv ci} db &= \left[(r + \delta - \alpha_c) b - \frac{1}{2} \alpha p k^{\alpha - 1} \right] dt + \sigma_c b dz_c \end{aligned} \tag{J.3.12}$$

with $b \equiv ci$. Finally, we get the stochastic differential equation describing the evolution of investments i by applying Ito's Lemma (10.2.2) on the function $J(b, c) = b/c$. Together with the stochastic differential equations (J.3.3) and (J.3.12) we get

$$\begin{aligned}
di = dJ(b,c) &= \underbrace{J_b}_{\frac{1}{c}} \left\{ \left[(r+\delta)b - \frac{1}{2}\alpha p_t k^{\alpha-1} \right] dt + \sigma_c b dz_c \right\} \\
&+ \underbrace{J_c}_{-\frac{b}{c^2}} \{\alpha_c c dt + \sigma_c c dz_c\} \\
&+ \frac{1}{2} \left[\underbrace{J_{bb}}_{=0} \sigma_c^2 b^2 + \overbrace{2 \underbrace{J_{bc}}_{=-\frac{1}{c^2}} \sigma_c^2 bc + \underbrace{J_{cc}}_{=\frac{b}{c^3}} \sigma_c^2 c^2}^{=-\sigma_c^2 \frac{b}{c}} \right] dt \\
&= \left[(r+\delta-\sigma_c^2-\alpha_c) \underbrace{\frac{b}{c}}_{=i} - \frac{1}{2c}\alpha p_t k^{\alpha-1} \right] dt - \sigma_c \frac{b}{c} dz_c + \sigma_c \frac{b}{c} dz_c \\
&= \left[(r+\delta-\sigma_c^2-\alpha_c) i - \frac{\alpha p_t}{2ck^{1-\alpha}} \right] dt
\end{aligned}$$

Together with the stochastic differential equations (J.3.2) and (J.3.3) we can completely describe the investment behavior i.

Exercise 3 b)

Same principle as in a) only that $\alpha_c = \sigma_c = 0$ now.

Exercise 3 c)

This is a long derivation but it follows the same principles as above, combining Brownian motion with Poisson jumps.

Exercise 3 d)

For this exercise we assume $\alpha_c = \sigma_c = 0$. Therefore the consumption is $c(\tau) = c_t$ in each point in time τ. Furthermore we assume that z_p is a Poisson process. Now we can solve the maximization problem (J.3.1) - (J.3.5) by using dynamic programming.

- DP 1: Bellman equation and first-order condition

In a first step we have to determine the Bellman equation. Therefore we use the CVF (10.2.4), thus we yield the Bellman equation

$$rV(k,p) = \max_{i(t)} \left\{ \begin{array}{l} pk^{\alpha} - ci^2 + V_k [i - \delta k] + V_p \alpha_p p \\ +\lambda [V(k,(1+\sigma_p)p) - V(k,p)] \end{array} \right\} \tag{J.3.13}$$

As before, the first-order condition of the Bellman equation (J.3.13) has the form

$$V_k = 2ci. \tag{J.3.14}$$

- DP 2: Evolution of the costate variable

The derivative of the maximized Bellman equation (J.3.13) with respect to the state variable k gives (using the Envelope theorem), the equation

$$rV_k = \frac{\alpha p}{k^{1-\alpha}} + V_{kk}(i - \delta k) - \delta V_k + \alpha_p p V_{kp} + \lambda \left[\tilde{V}_k - V_k \right] \tag{J.3.15}$$

Computing the differential of the shadow price of capital V_k gives, using the CVF (10.2.4)

$$dV_k = (i - \delta k) V_{kk} dt + \alpha_p p V_{kp} dt + \left[\tilde{V}_k - V_k \right] dq \tag{J.3.16}$$

with $\tilde{V}_k = V(k, p(1+\alpha_p))$.

If we insert (J.3.15) into (J.3.16) we get

$$dV_k = \left\{ (r+\delta) V_k - \frac{\alpha p}{k^{1-\alpha}} - \lambda \left[\tilde{V}_k - V_k \right] \right\} dt + \left[\tilde{V}_k - V_k \right] dq \tag{J.3.17}$$

- DP 3: Inserting the first-order condition

Now we replace V_k and $\tilde{V}_k$ by (J.3.14) in equation (J.3.17). Therefore we get

$$\begin{aligned} 2cdi &= \left\{ 2c(r+\delta)i - \frac{\alpha p}{k^{1-\alpha}} - 2c\lambda[\tilde{\imath} - i] \right\} dt + 2c[\tilde{\imath} - i]dq \\ \Leftrightarrow di &= \left\{ i(r+\delta) - \frac{\alpha p}{2ck^{1-\alpha}} - \lambda[\tilde{\imath} - i] \right\} dt + [\tilde{\imath} - i]dq \end{aligned}$$

Together with the SDEs (J.3.2) and (J.3.4) we can completely describe the investment behavior i.

Exercise 4 a)

What is the optimal static employment level l for a given technological level q? The demand function is $x = \Phi p^{-\varepsilon} \Leftrightarrow p = \left(\frac{\Phi}{x}\right)^{\frac{1}{\varepsilon}}$, the technology is given by $x = a^q l, \quad a > 1$ and the wage rate is w.

Profits are then given by $\pi(l) = a^q l \left(\frac{\Phi}{a^q l}\right)^{\frac{1}{\varepsilon}} - wl$ and the first-order condition with respect to employment reads

$$\frac{a^q(\varepsilon - 1)\left(\frac{\Phi}{a^q l}\right)^{\frac{1}{\varepsilon}}}{\varepsilon} - w = 0$$

The optimal static employment level is therefore

$$l^* = \frac{\Phi\left(\frac{w\cdot\varepsilon}{a^q(\varepsilon-1)}\right)^{-\varepsilon}}{a^q}.$$

Output and price are then

$$x^* = \Phi\left(\frac{w/a^q}{(\varepsilon-1)/\varepsilon}\right)^{-\varepsilon} \text{ and } p^* = \frac{w\varepsilon}{a^q(\varepsilon-1)} = \frac{\frac{w}{a^q}}{\frac{\varepsilon-1}{\varepsilon}}.$$

The last ratio simply divides marginal costs by the mark-up parameter.

Exercise 4 b)

The intertemporal objective function reads $\Pi_t = \int_t^\infty e^{-\rho[\tau-t]}\pi(l(\tau))d\tau$. Since q is constant over time, the optimal employment level is constant as well and given by previous solution

$$l^*(\tau) = \frac{\Phi\left(\frac{w\cdot\varepsilon}{a^q(\varepsilon-1)}\right)^{-\varepsilon}}{a^q} \quad \forall\tau \geq t.$$

The result does therefore not change with respect to (a).

Exercise 4 c)

We now determine both l and l_q optimally. The objective function now reads

$$\Pi_t = \int_t^\infty e^{-\rho[\tau-t]}\pi(l(\tau), l_q(\tau))d\tau$$

with instantaneous profits being given by

$$\pi(l(\tau), l_q(\tau)) = a^{q(\tau)}l(\tau)\left(\frac{\Phi}{a^{q(\tau)}l(\tau)}\right)^{\frac{1}{\varepsilon}} - w[l(\tau) + l_q(\tau)]$$

and $q(\tau)$ being a Poisson Process with arrival rate $\lambda(l_q)$.

- DP 1: Bellman equation and first-order conditions

The optimal programme is $V(q(t)) \equiv \max_{\{l(\tau),l_q(\tau)\}} \Pi(t)$ subject to the constraints. The general Bellman equation reads

$$\rho V(q(t)) = \max_{l(t),l_q(t)} \left\{ \pi(l(t), l_q(t)) + \frac{1}{dt} E_t dV((q(t)) \right\}.$$

With the differential $dV(q(t)) = \{V(q+1) - V(q)\}dq$ and after forming expectations,

$$E_t dV(q(t)) = \{V(q+1) - V(q)\}E_t dq = \{V(q+1) - V(q)\}\lambda(l_q)dt,$$

the specific Bellman equation is

$$\rho V(q(t)) = \max_{l(t),l_q(t)} \{\pi(l(t), l_q(t)) + \lambda(l_q)[V(q+1) - V(q)]\}.$$

The first-order conditions are

$$\begin{aligned} \pi_l &= \frac{a^q(\varepsilon-1)\left(\frac{\Phi}{a^q l}\right)^{\frac{1}{\varepsilon}}}{\varepsilon} - w = 0 \Leftrightarrow l(t) = \frac{\Phi\left(\frac{w \cdot \varepsilon}{a^{q(t)}(\varepsilon-1)}\right)^{-\varepsilon}}{a^{q(t)}}, \\ w &= \lambda_{l_q}[V(q+1) - V(q)]. \end{aligned}$$

Exercise 4 d)

We now compute the expected output level for $\tau > t$. We can write output in τ as

$$x(\tau) = aq(\tau)\, l(\tau) = aq(\tau) \frac{\Phi\left(\frac{w\varepsilon}{aq(\tau)(\varepsilon-1)}\right)^{-\varepsilon}}{aq(\tau)} = \Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} a^{\varepsilon q_\tau}.$$

Its mean is

$$\begin{aligned} E_t x(\tau) &= E_t\left(\Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} a^{\varepsilon q(\tau)}\right) = \Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} E_t(a^{\varepsilon q(\tau)}) \\ &\overset{\text{Lemma 8 (section 9.4.1)}}{=} \Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} a^{\varepsilon q(t)} e^{(a^{\varepsilon}-1)\lambda(l_q^*)(\tau-t)},\ \tau > t, \end{aligned}$$

where the last step assumed a constant arrival rate.

End of solutions manual

For references and index, please see page 299 and onwards.