Klaus Wälde

Applied Intertemporal Optimization

Edition 1.2 plus: Textbook and Solutions Manual

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Klaus Wälde

Gutenberg School of Management and Economics
Gutenberg Research College Mainz
Gutenberg University Mainz
Mainz
Germany

www.waelde.com

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Acknowledgments

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I also profited from many discussions with economists and mathematicians from many other places. The high download figures at repec.org show that the book is also widely used on an international level. I especially would like to thank Christian Bayer and Ken Sennewald for many insights into the more subtle issues of stochastic continuous time processes. I hope I succeeded in incorporating these insights into this book. I would also like to thank MacKichan for their repeated, quick and very useful support on typesetting issues in Scientific Word.

Edition 1.1 from 2011 is almost identical from the first edition in 2010. Some typos were removed and few explanations were improved. Edition 1.2 exists only as edition 1.2 plus: Textbook and Solutions Manual. Apart from some minor changes as compared to edition 1.1., it includes solutions to selected problem sets as described at the beginning of the Solutions Manual on p. 321.

Mainz, August 2012

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Part V

Solutions manual

Appendix A

Introduction to solutions to exercises

This solutions manual presents solutions to selected exercises. There are not solutions to all exercises simply as not all of the existing solutions are nicely typed, edited and checked for errors at this point. There are solutions to the following exercises.

Chapter 1 – no exercises
Chapter 2 – Exercises 2, 8 and 10
Chapter 3 – Ex. 2, 5 and 7
Chapter 4 – 1, 2, 6, 8 and 9
Chapter 5 – 1, 4, 5 and 8
Chapter 6 – 1
Chapters 7 and 8 – 6
Chapter 9 – 2, 7, 8 and 10
Chapter 10 – 3, 4, 5, 12 and 13
Chapter 11 – 1, 2 and 3

Any book goes through various stages before actually being "born" in its printed version. This is also true for this book. It started with simple notes, went through its adolescence as various informal versions for the internet and became grown-up via the incorporation of serious and thoughtful comments by referees.

I should probably mention that the solutions manual is not as well-tested as the text itself. While I believe that the solutions are correct, the ideas behind them are not explained in an as elaborated way as in the main text – after all, it is a manual. I am nevertheless sure that it will be very useful for users of this book. As always, any comments are welcome.

The main text has been written word for word by the author of the book. The solutions of the exercises were all suggested by the author. A large group of collaborators have contributed, edited and refined them. They include Ken Sennewald, Christopher Kops and Michael Lamprecht. I am highly grateful to them for their support.

Appendix B

Solution to exercises of chapter 2

B.1 Solving by substitution – ex. 2

From the main text (see equations (2.2.1), (2.1.2) and (2.1.3)) we already know the following equations

$$\text{objective function:} \qquad U_t = \gamma \ln c_t + (1-\gamma)\ln c_{t+1} \tag{B.1.1}$$

$$\text{budget constraint in } t\text{:} \qquad c_t = w_t - s_t \tag{B.1.2}$$

$$\text{budget constraint in } t+1\text{:} \qquad c_{t+1} = w_{t+1} + (1+r_{t+1})s_t. \tag{B.1.3}$$

Now the principle is to transform this optimization problem with constraints into one without constraints. We therefore insert the budget constraints (B.1.2) and (B.1.3) into our objective function (B.1.1). The unconstrained maximization problem then reads $\max U_t$ by choosing s_t, where

$$U_t = \gamma \ln(w_t - s_t) + (1-\gamma)\ln(w_{t+1} + (1+r_{t+1})s_t).$$

The derivative with respect to savings s_t is given by

$$\frac{\gamma}{w_t - s_t} = (1+r_{t+1})\frac{1-\gamma}{w_{t+1} + (1+r_{t+1})s_t},$$

which represents the first-order condition. When this is solved for savings s_t, we obtain

$$\frac{w_{t+1}}{1+r_{t+1}} + s_t = (w_t - s_t)\frac{1-\gamma}{\gamma} \Leftrightarrow \frac{w_{t+1}}{1+r_{t+1}} = w_t\frac{1-\gamma}{\gamma} - s_t\frac{1}{\gamma} \Leftrightarrow$$

$$s_t = (1-\gamma)w_t - \gamma\frac{w_{t+1}}{1+r_{t+1}} = w_t - \gamma W_t,$$

where W_t is life-time income as defined after (2.2.3).

To show that this is the same result as in (2.2.4) and (2.2.5) we compute first- and second-period consumption and find

$$c_t = w_t - s_t = \gamma W_t$$

$$c_{t+1} = w_{t+1} + (1+r_{t+1})s_t = w_{t+1} + (1+r_{t+1})(w_t - \gamma W_t) = (1+r_{t+1})(1-\gamma)W_t,$$

where the last equation used again the definition (2.2.3) of life-time income W_t.

- DP 1: Bellman equation and first-order conditions

The optimal programme is $V(q(t)) \equiv \max_{\{l(\tau),l_q(\tau)\}} \Pi(t)$ subject to the constraints. The general Bellman equation reads

$$\rho V(q(t)) = \max_{l(t),l_q(t)} \left\{ \pi(l(t), l_q(t)) + \frac{1}{dt} E_t dV((q(t)) \right\}.$$

With the differential $dV(q(t)) = \{V(q+1) - V(q)\}dq$ and after forming expectations,

$$E_t dV(q(t)) = \{V(q+1) - V(q)\}E_t dq = \{V(q+1) - V(q)\}\lambda(l_q)dt,$$

the specific Bellman equation is

$$\rho V(q(t)) = \max_{l(t),l_q(t)} \{\pi(l(t), l_q(t)) + \lambda(l_q)[V(q+1) - V(q)]\}.$$

The first-order conditions are

$$\begin{aligned} \pi_l &= \frac{a^q(\varepsilon-1)\left(\frac{\Phi}{a^q l}\right)^{\frac{1}{\varepsilon}}}{\varepsilon} - w = 0 \Leftrightarrow l(t) = \frac{\Phi\left(\frac{w\cdot\varepsilon}{a^{q(t)}(\varepsilon-1)}\right)^{-\varepsilon}}{a^{q(t)}}, \\ w &= \lambda_{l_q}[V(q+1) - V(q)]. \end{aligned}$$

Exercise 4 d)

We now compute the expected output level for $\tau > t$. We can write output in τ as

$$x(\tau) = aq(\tau)\, l(\tau) = aq(\tau) \frac{\Phi\left(\frac{w\varepsilon}{aq(\tau)(\varepsilon-1)}\right)^{-\varepsilon}}{aq(\tau)} = \Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} a^{\varepsilon q_\tau}.$$

Its mean is

$$\begin{aligned} E_t x(\tau) &= E_t\left(\Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} a^{\varepsilon q(\tau)}\right) = \Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} E_t(a^{\varepsilon q(\tau)}) \\ &\overset{\text{Lemma 8 (section 9.4.1)}}{=} \Phi\left(\frac{\varepsilon-1}{w\varepsilon}\right)^{\varepsilon} a^{\varepsilon q(t)} e^{(a^\varepsilon-1)\lambda(l_q^*)(\tau-t)},\ \tau > t, \end{aligned}$$

where the last step assumed a constant arrival rate.

End of solutions manual

For references and index, please see page 299 and onwards.