5 Appendix

Referees' appendix to "Production technologies in stochastic continuous time models" by Klaus Wälde

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5.1 Differential technology

5.1.1 Deriving the maximized BE

The optimal consumption path can be found by maximizing equation (1) with respect to C taking the equation for the capital accumulation (12) into account. This can be done by using the dynamic programming approach according to Bellman

$$\rho V(K) = \max_C \left[u(C) + \frac{E[dV(K)]}{dt} \right],$$

where because of Ito's Formula for Ito-Lévy processes (cf. e.g. Øksendal and Sulem (2005), Theorem 1.16)

$$dV(K) = V_K((A-\delta)K - C)dt - V_K \sigma K dz + \frac{1}{2} V_{KK} \sigma^2 K^2 dt + \Sigma_{i=1}^n (V(K-\beta_i K) - V(K)) dq_i.$$

Bearing the martingale property of both the the term involving the Brownian motion process dz and the terms involving the Poisson processes dq_i in mind, i.e. $E[V_K \sigma K dz] = 0$ and $E[(V(K-\beta_i K) - V(K)) dq_i] = \lambda_i (V(K-\beta_i K) - V(K)) dt$, one obtains:

$$\frac{E[dV(K)]}{dt} = V_K((A-\delta)K - C) + \frac{1}{2}\sigma^2 K^2 V_{KK} + \Sigma_{i=1}^n \lambda_i (V(K-\beta_i K) - V(K))$$

Inserting this into the BE one gets:

$$\rho V(K) = \max_C \left[u(C) + V_K((A-\delta)K - C) + \frac{1}{2}\sigma^2 K^2 V_{KK} + \Sigma_{i=1}^n \lambda_i (V(K-\beta_i K) - V(K)) \right] \tag{24}$$

Maximizing with respect to C delivers the following first order condition (FOC)

$$u_C \equiv V_K,$$

which can be because of (2) written as

$$C^* = \frac{1}{V_K}. \tag{25}$$

Plugging this optimum condition into (24) gives the maximized BE (13).

5.1.2 Deriving C under log-utility

Computing the first and the second derivative of the educated guess (14) with respect to K gives:

$$V_K = \frac{1}{\psi K} \tag{26}$$

$$V_{KK} = -\frac{1}{\psi K^2} \tag{27}$$

Inserting these two expressions as well as

$$\Sigma_{i=1}^{n} \lambda_i \left(V \left(K - \beta_i K \right) - V \left(K \right) \right) = \frac{1}{\psi} \Sigma_{i=1}^{n} \lambda_i \log \left(1 - \beta_i \right)$$

into the maximized BE (13) yields:

$$0 = \log \left(\psi K \right) - \rho \left(\frac{\log \left(\psi K \right)}{\psi} + \Psi \right) + \frac{1}{\psi K} \left(\left(A - \delta \right) K - \psi K \right) - \frac{1}{2} \frac{\sigma^2}{\psi} + \frac{1}{\psi} \Sigma_{i=1}^{n} \lambda_i \log \left(1 - \beta_i \right)$$
$$= \log \left(\psi K \right) - \frac{\rho \log \left(\psi K \right)}{\psi} - \rho \Psi + \frac{A - \delta - \psi - \frac{1}{2} \sigma^2 + \Sigma_{i=1}^{n} \lambda_i \log \left(1 - \beta_i \right)}{\psi}$$

Choosing

$$\rho = \psi \tag{28}$$

and

$$\Psi = \frac{A - \delta - \rho - \frac{1}{2} \sigma^2 + \Sigma_{i=1}^{n} \lambda_i \log \left(1 - \beta_i \right)}{\rho^2}$$

one can verify that the educated guess for the value function V is correct. By combining the FOC (25) with (26) and (28) one can replicate equation (15).

5.1.3 Deriving C under CRRA utility

If instantaneous utility is given by equation (2) the structure of the BE remains unchanged compared to (24) but the resulting FOC looks different:

$$C^* = V_K^{-\frac{1}{\gamma}} \tag{29}$$

This expression leads to the following maximized BE:

$$0 = \frac{\left(V_K^{-\frac{1}{\gamma}} \right)^{1-\gamma} - 1}{1 - \gamma} - \rho V \left(K \right) + V_K \left(\left(A - \delta \right) K - V_K^{-\frac{1}{\gamma}} \right) +$$
$$\frac{1}{2} \sigma^2 K^2 V_{KK} + \Sigma_{i=1}^{n} \lambda_i \left(V \left(K - \beta_i K \right) - V \left(K \right) \right)$$

Once again one has to make an educated guess concerning the value function V:

$$V \left(K \right) \equiv \frac{\psi^{-\gamma} K^{1-\gamma}}{1 - \gamma} + \Psi$$

Inserting the first and the second derivative

$$V_K = \psi^{-\gamma} K^{-\gamma} \tag{30}$$

$$V_{KK} = -\psi^{-\gamma} \gamma K^{-\gamma-1} \tag{31}$$

as well as

$$\Sigma_{i=1}^{n} \lambda_i \left(V \left(K - \beta_i K \right) - V \left(K \right) \right) = \psi^{-\gamma} \frac{K^{1-\gamma}}{1-\gamma} \Sigma_{i=1}^{n} \lambda_i \left(\left(1 - \beta_i \right)^{1-\gamma} - 1 \right)$$

into the maximized BE gives:

$$0 = \frac{\left(\left(\psi^{-\gamma} K^{-\gamma} \right)^{-\frac{1}{\gamma}} \right)^{1-\gamma} - 1}{1-\gamma} - \rho \frac{\psi^{-\gamma} K^{1-\gamma}}{1-\gamma} - \rho \Psi + \psi^{-\gamma} K^{-\gamma} \left(\left(A - \delta \right) K - \left(\psi^{-\gamma} K^{-\gamma} \right)^{-\frac{1}{\gamma}} \right) -$$
$$\frac{1}{2} \sigma^2 K^2 \psi^{-\gamma} \gamma K^{-\gamma-1} + \psi^{-\gamma} \frac{K^{1-\gamma}}{1-\gamma} \Sigma_{i=1}^{n} \lambda_i \left(\left(1 - \beta_i \right)^{1-\gamma} - 1 \right)$$
$$= \frac{\psi^{1-\gamma}}{1-\gamma} - \frac{1}{\left(1-\gamma \right) K^{1-\gamma}} - \rho \frac{\psi^{-\gamma}}{1-\gamma} - \frac{\rho \Psi}{K^{1-\gamma}} + \psi^{-\gamma} \left(\left(A - \delta \right) - \psi \right) -$$
$$\frac{1}{2} \sigma^2 \psi^{-\gamma} \gamma + \psi^{-\gamma} \frac{1}{1-\gamma} \Sigma_{i=1}^{n} \lambda_i \left(\left(1 - \beta_i \right)^{1-\gamma} - 1 \right)$$

In order to guarantee that this equation holds the constant Ψ has to be

$$\Psi = -\frac{1}{\left(1-\gamma \right) \rho}$$

and ψ has to be as given in equation (10). For $\gamma = 1$, i.e. the case of log-utility, this equation collapses to the result that $\psi = \rho$ which just is expression (28). The relationship (10) together with (29) and (30) produce equation (16).

5.2 Standard I technology

5.2.1 Expected growth of output

By taking equation (32) and plugging the finding of Section 3.2.1 that optimal consumption is the constant fraction ρ of the capital stock (cf. eq. (22)) into equation (8) one can derive via Ito's Formula for Ito-Lévy processes (cf. e.g. Øksendal and Sulem (2005), Theorem 1.14) for the production function (6) that the evolution of output has to obey:

$$dY \left(t \right) = \left(A \left(t \right) - \delta - \rho + \mu \right) Y \left(t \right) dt + \sigma Y \left(t \right) dz \left(t \right) + \Sigma_{i=1}^{n} \beta_i Y \left(t \right) dq_i \left(t \right)$$

Taking expectations and dividing by dt as well as $Y(t)$ deliver the expression for the expected growth rate of output

$$\frac{E_t \left[\frac{dY(t)}{st} \right]}{Y \left(t \right)} = A \left(t \right) - \delta - \rho + \mu + \Sigma_{i=1}^{n} \beta_i \lambda_i,$$

which depends on $A(t)$.

5.2.2 Definition of A as a martingale

Inserting (11) in (7) delivers

$$dA(t) = \mu A(t)\,dt + \sigma A(t)\,dz(t) + \Sigma_{i=1}^{n}\beta_i A(t)\,dq_i(t) \tag{32}$$

or in integral notation:

$$A(\tau) - A(t) = \mu \int_t^{\tau} A(s)\,ds + \sigma \int_t^{\tau} A(s)\,dz(s) + \Sigma_{i=1}^{n}\beta_i \int_t^{\tau} A(s)\,dq_i(s)$$

Taking expectations on both sides of this equation gives

$$E_t[A(\tau) - A(t)] = \mu E_t\left[\int_t^{\tau} A(s)\,ds\right] + \sigma E_t\left[\int_t^{\tau} A(s)\,dz(s)\right] + \Sigma_{i=1}^{n}\beta_i E_t\left[\int_t^{\tau} A(s)\,dq_i\right],$$

where the second term on the right hand side of the equation is equal to zero. Using Fubini's theorem one can therefore write the last equation as

$$E_t[A(\tau)] - A(t) = \mu \int_t^{\tau} E_t[A(s)]\,ds + \Sigma_{i=1}^{n}\beta_i\lambda_i \int_t^{\tau} E_t[A(s)]\,ds.$$

After having taken the derivative with respect to τ employing the Leibniz rule one ends up with the following ODE

$$\frac{dE_t[A(\tau)]}{d\tau} = \mu E_t[A(\tau)] + \Sigma_{i=1}^{n}\beta_i\lambda_i E_t[A(\tau)] = \left(\mu + \Sigma_{i=1}^{n}\beta_i\lambda_i\right) E_t[A(\tau)],$$

that can be solved for the expectation value of $A(\tau)$:

$$E_t[A(\tau)] = A(t)\exp\left(\left(\mu + \Sigma_{i=1}^{n}\beta_i\lambda_i\right)(\tau - t)\right) \tag{33}$$

5.2.3 Deriving the maximized BE

The adequate BE corresponding to the maximization problem of the social planner reads:

$$\rho V(A,K) = \max_C \left[u(C) + \frac{E[dV(A,K)]}{dt}\right]$$

Applying Ito's Formula to the term $dV(A,K)$ delivers the following output:

$$\begin{gathered} dV(A,K) = V_K(AK - \delta K - C)\,dt + V_A A\mu dt + \frac{1}{2}V_{AA}A^2\sigma^2 dt + \\ V_A A\sigma dz + \Sigma_{i=1}^{n}\left(V(A + A\beta_i, K) - V(A,K)\right) dq_i \end{gathered}$$

Taking expectations, dividing by dt, and inserting the resulting expression into the BE gives:

$$\rho V(A,K) = \max_C \left[\begin{array}{c} u(C) + V_K(AK - \delta K - C) + V_A A\mu + \frac{1}{2}V_{AA}A^2\sigma^2 + \\ \Sigma_{i=1}^{n}\lambda_i\left(V(A + A\beta_i, K) - V(A,K)\right) \end{array}\right] \tag{34}$$

Maximizing with respect to C produces the following first order condition (FOC)

$$u_C \equiv V_K,$$

which can be because of (2) written as

$$C^* = \frac{1}{V_K}. \tag{35}$$

Plugging this optimum condition into (34) gives the maximized BE (18).

5.2.4 Deriving C under log-utility

Inserting the educated guess for C (cf. eq. (19)) into equation (8) one can find $K(\tau)$ to be:

$$K(\tau) = K(t)\exp\left(\int_t^\tau \left(A(s) - \delta - \tilde{\psi}\right) ds\right) \tag{36}$$

Furthermore the solution to (32) looks like[7]

$$A(\tau) = A(t)\exp\left(\mu(\tau - t) - \frac{1}{2}\sigma^2(\tau - t) + \sigma z(\tau) + \log\left(\Sigma_{i=1}^n \frac{1+\beta_i}{n}\right)\Sigma_{i=1}^n \frac{q_i(\tau)}{n}\right). \tag{37}$$

Recognizing that the variable $\exp(\sigma z(\tau))$ is log-normal distributed with $E_t[\exp(\sigma z(\tau))] = \exp\left(\frac{1}{2}\sigma^2(\tau - t)\right)$ allows to derive a solution for $E_t\left[\exp\left(\log\left(\Sigma_{i=1}^n\left(\frac{1+\beta_i}{n}\right)\right)\Sigma_{i=1}^n\frac{q_i(\tau)}{n}\right)\right]$ by taking expectations of the relationship (37) and comparing the resulting expression with (33):

$$\begin{aligned}
E_t[A(\tau)] &= E_t\left[A(t)\exp\left(\mu(\tau - t) - \frac{1}{2}\sigma^2(\tau - t) + \sigma z(\tau) + \log\left(\Sigma_{i=1}^n\frac{1+\beta_i}{n}\right)\Sigma_{i=1}^n\frac{q_i(\tau)}{n}\right)\right] \\
&= A(t)\exp(\mu(\tau - t))\exp\left(-\frac{1}{2}\sigma^2(\tau - t)\right)E_t[\exp(\sigma z(\tau))] \\
&\quad E_t\left[\exp\left(\Sigma_{i=1}^n \log\left(\Sigma_{i=1}^n\frac{1+\beta_i}{n}\right)\Sigma_{i=1}^n\frac{q_i(\tau)}{n}\right)\right] \\
&= A(t)\exp\left(\left(\mu + \Sigma_{i=1}^n\beta_i\lambda_i\right)(\tau - t)\right)
\end{aligned}$$

$$\Longleftrightarrow E_t\left[\exp\left(\log\left(\Sigma_{i=1}^n\left(\frac{1+\beta_i}{n}\right)\right)\Sigma_{i=1}^n\frac{q_i(\tau)}{n}\right)\right] = \exp\left(\Sigma_{i=1}^n\beta_i\lambda_i(\tau - t)\right) \tag{38}$$

Inserting this as well as equations (2), (19), (36), and (37) into the definition of the value function $V_t(A,K) \equiv \max_{\{C(\tau)\}} U_t$ gives

$$\begin{aligned}
V_t(A,K) &= E_t\left[\int_t^\infty \exp(-\rho(\tau - t))\, u(C^*(\tau))\, d\tau\right] = E_t\left[\int_t^\infty \exp(-\rho(\tau - t))\, u\left(\tilde{\psi}K(\tau)\right) d\tau\right] \\
&= E_t\left[\int_t^\infty \exp(-\rho(\tau - t))\log\left(\tilde{\psi}K(t)\exp(\Phi)\right) d\tau\right],
\end{aligned}$$

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[7] This can be proven by applying Ito's Formula

$$\begin{aligned}
dA(\tau) &= A(\tau)\mu d\tau - A(\tau)\frac{1}{2}\sigma^2 d\tau + A(\tau)\sigma dz(\tau) + \frac{1}{2}A(\tau)\sigma^2(dz(\tau))^2 + \\
&\quad \textstyle\sum_{i=1}^n (A(\tau,z,q_i+1) - A(\tau,z,q_i))\, dq_i(\tau) \\
&= A(\tau)\mu d\tau + A(\tau)\sigma dz(\tau) + \\
&\quad \textstyle\sum_{i=1}^n \left(A(t)\exp\left(\begin{array}{c}\mu(\tau - t) - \frac{1}{2}\sigma^2(\tau - t) + \sigma z(\tau) + \\ \log\left(\sum_{i=1}^n\frac{1+\beta_i}{n}\right)\left(\sum_{i=1}^n\frac{q_i(\tau)+1}{n}\right)\end{array}\right) - A(\tau)\right) dq_i(\tau) \\
&= A(\tau)\mu d\tau + A(\tau)\sigma dz(\tau) + \textstyle\sum_{i=1}^n\left(A(\tau)\sum_{i=1}^n\frac{1+\beta_i}{n} - A(\tau)\right) \\
&= \mu A(\tau)d\tau + \sigma A(\tau)dz(\tau) + \textstyle\sum_{i=1}^n\beta_i A(\tau)\, dq_i(\tau),
\end{aligned}$$

where the last expression is identical to equation (32).

where

$$\Phi \equiv \int_t^\tau \left(A(t) \exp \begin{pmatrix} \mu (s-t) - \frac{1}{2}\sigma^2 (s-t) + \sigma z(s) + \\ \log \left(\Sigma_{i=1}^n \frac{1+\beta_i}{n} \right) \Sigma_{i=1}^n \frac{q_i(s)}{n} \end{pmatrix} - \delta - \tilde{\psi} \right) ds$$

which is expression (20) in the text. Simplifying and taking the condition (17) as well as the expectation value (38) into account yields:

$$\begin{aligned}
V_t(A,K) &= \log\left(\tilde{\psi}K(t)\right) \int_t^\infty \exp(-\rho(\tau - t))\, d\tau - \left(\delta + \tilde{\psi}\right) \int_t^\infty \exp(-\rho(\tau-t))(\tau - t)\, d\tau + \\
&\quad A(t) \int_t^\infty \exp(-\rho(\tau - t)) \\
&\quad \int_t^\tau \begin{pmatrix} \exp(\mu(s-t)) \exp\left(-\frac{1}{2}\sigma^2(s-t)\right) E_t\left[\exp(\sigma z(s))\right] \\ E_t\left[\exp\left(\log\left(\Sigma_{i=1}^n \frac{1+\beta_i}{n}\right) \Sigma_{i=1}^n \frac{q_i(s)}{n}\right)\right] \end{pmatrix} ds d\tau \\
&= \log\left(\tilde{\psi}K(t)\right) \left[\frac{\exp(-\rho(\tau-t))}{-\rho}\right]_t^\infty - \left(\delta + \tilde{\psi}\right) \left[\frac{\exp(-\rho(\tau-t))}{(-\rho)^2}(-\rho(\tau-t)-1)\right]_t^\infty + \\
&\quad A(t) \left[\frac{\exp(-\rho(\tau-t))}{(-\rho)^2}(-\rho(\tau-t)-1)\right]_0^\infty \\
&= \frac{\log\left(\tilde{\psi}K(t)\right)}{\rho} + \frac{A(t)}{\rho^2} - \frac{\delta + \tilde{\psi}}{\rho^2}
\end{aligned}$$

As this has to hold irrespective of time t one can drop the time index and obtain the value function (21). It follows that

$$V_K = \frac{1}{\rho K},$$

$$V_A = \frac{1}{\rho^2},$$

$$V_{AA} = 0,$$

and

$$\Sigma_{i=1}^n \lambda_i \left(V(A + A\beta_i, K) - V(A,K)\right) = \frac{1}{\rho^2} A \Sigma_{i=1}^n \lambda_i \beta_i.$$

Inserting all these expressions into the maximized BE (18) delivers

$$\begin{aligned}
\rho\left(\frac{\log\left(\tilde{\psi}K\right)}{\rho} + \frac{A}{\rho^2} - \frac{\delta+\tilde{\psi}}{\rho^2}\right) &= \log(\rho K) + \frac{1}{\rho K}(AK - \delta K - \rho K) + \frac{1}{\rho^2} A\mu + \frac{1}{\rho^2} A \Sigma_{i=1}^n \lambda_i \beta_i \\
&= \log(\rho K) + \frac{A}{\rho} - \frac{\delta}{\rho} - 1,
\end{aligned}$$

where the second equality holds because of equation (17). Again, one can see that for $\tilde{\psi} = \rho$ this equation is satisfied and therefore like in the differential setup optimal consumption is indeed a constant fraction of the capital stock (cf. eq. (22)).